Math 307, Spring 2012

Midterm 2B Wednesday, May 16th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:________________________

1	8	
2	32	
3	10	
Total	50	

- Please check that your exam contains 3 problems on 6 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

GOOD LUCK!

1. (2 points each; 8 points total) Suppose you are given a forced, damped spring-mass system that satisfies the equation $mu'' + \gamma u' + ku = F_0 \cos \omega t$, with $u(0) = u_0$ and $u'(0) = v_0$. Circle or write TRUE or FALSE.

(a) TRUE or FALSE: Given specific values for $m, \gamma, k, F_0, \omega, u_0$, and v_0, there is always a solution.

we know how to find it!

(b) TRUE or FALSE: For some values of the constants, the amplitude of the forced response can be larger than F_0/k. (Recall that a static force of F_0 would stretch the spring by a distance F_0/k.)

near resonance.

(c) TRUE or FALSE: At $t = 0$, the value of the transient solution can be greater than the ~~amplitude~~ of the steady state solution.

mathematically the two are unrelated

(d) TRUE or FALSE: Resonance can happen with or without damping.

we discussed ~~sta~~ resonance in both situations (§3.7 and §3.8)

2. (4 points each; 32 points total) For parts (a) through (g), find the general solution for the given differential equation.

(a) $y'' - 2y = 0$.

characteristic equation: $r^2 - 2 = 0$

roots: $r = \pm\sqrt{2}$

(2 pts)

$$\boxed{y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}}$$

(2 pts)

(b) $y'' - 2y' = 0$.

$r^2 - 2r = 0$

$r(r-2) = 0$

$r = 0, 2$

$$\boxed{y(t) = c_1 + c_2 e^{2t}}$$

(c) $y'' + y' + y = 0$.

$r^2 + r + 1 = 0$

$r = \frac{-1 \pm \sqrt{1 - 4\cdot 1\cdot 1}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$

$$\boxed{y(t) = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)}$$

(d) $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1$ repeated root

$$\boxed{y(t) = c_1 e^{t} + c_2 t e^{t}}$$

(e) $y'' - 2y' + y = -e^t$. $\quad y = y_c + Y$

for homogeneous, see (d).

$y_c(t) = c_1 e^t + c_2 t e^t$

For particular soln, both Ae^t and Ate^t are sol'ns to homogeneous, so

guess: $Y = At^2 e^t$

then $Y' = At^2 e^t + 2Ate^t$

$Y'' = At^2 e^t + 4Ate^t + 2Ae^t$

plug into $y'' - 2y' + y = -e^t$

get

$\underbrace{(A - 2A + A)}_{\text{zero}} t^2 e^t + \underbrace{(4A - 4A)}_{\text{zero}} te^t + 2Ae^t = -e^t$

$2A = -1$

$A = -\frac{1}{2}$

general solution:

$$\boxed{y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} t^2 e^t}$$

(f) $y'' + y = \sin t$. $\quad y = y_c + Y$

for homogeneous, $r^2 + 1 = 0$

$r = \pm i$

$y_c = c_1 \cos t + c_2 \sin t$

for particular,

guess: $Y = At\cos t + Bt\sin t$

$Y' = -At\sin t + A\cos t + Bt\cos t + B\sin t$

$Y'' = -At\cos t - A\sin t - A\sin t - Bt\sin t + B\cos t + B\cos t$

plug in to $y'' + y = \sin t$, get

$(-A + A)t\cos t + (B - B)t\sin t + (-2A\sin t) + 2B\cos t = \sin t$

$-2A = 1 \qquad 2B = 0$

$A = \frac{-1}{2} \qquad B = 0$

then general sol'n is

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2} t\cos t}$$

(g) $y'' + 2y = \sin 2t$. $y = y_c + Y$

For homogeneous, $r^2 + 2 = 0$

$r = \pm i\sqrt{2}$, so

$y_c(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t)$.

For particular, guess

$Y = A\cos 2t + B\sin 2t$

$Y' = -2A\sin 2t + 2B\cos 2t$

$Y'' = -4A\cos 2t - 4B\sin 2t$

plug into $y'' + 2y = \sin 2t$, get

$(-4A + 2A)\cos 2t + (-4B + 2B)\sin 2t = \sin 2t$

$-2A = 0$ $\quad$ $-2B = 1$

$A = 0$ $\quad$ $B = -\frac{1}{2}$

so general solution is

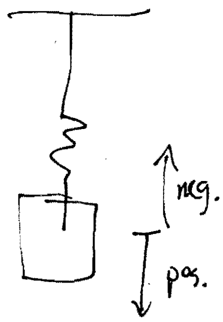
$$\boxed{y(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) - \frac{1}{2}\sin 2t}$$

(h) Write down what guess you would use for the particular solution to the following differential equation. DO NOT solve the equation.

$$y'' + 2y = e^t - t^2\cos 2t.$$

$$Y(t) = Ae^t + (Bt^2 + Ct + D)(E\cos 2t + F\sin 2t)$$

3. (10 points total) For this problem you're working with a spring with $k = 49$ N/m. Assume there is no damping. You can leave your answers in exact form.



(a) (7 points) A birthday cake of unknown mass hangs from the spring. It's pulled 25 cm down from equilibrium and set in motion with an upward velocity of 1 m/s. You measure the amplitude of the resulting oscillation to be 50 cm. What is the mass of the object? Include units.

$k = 49 \, N/m$ $\quad mu'' + 49u = 0$ $\quad u(0) = 25\text{ cm} = 0.25 \text{ m}$

$\gamma = 0$ $\quad mr^2 + 49 = 0$ $\quad u'(0) = -1 \text{ m/s}$

$r = \pm \frac{i \cdot 7}{\sqrt{m}}$ $\quad$ amplitude $= 0.5$ m

everything is in metric

general solution: $u(t) = c_1 \cos\left(\frac{7}{\sqrt{m}}t\right) + c_2 \sin\left(\frac{7}{\sqrt{m}}t\right)$ (+2pts)

plug in ICs: $u(0) = c_1 = 0.25$ (+1 pt)

$$u'(t) = -\frac{7}{\sqrt{m}} c_1 \sin\left(\frac{7}{\sqrt{m}}t\right) + \frac{7}{\sqrt{m}} c_2 \cos\left(\frac{7}{\sqrt{m}}t\right)$$

$$u'(0) = \frac{7}{\sqrt{m}} c_2 = -1 \longrightarrow c_2 = \frac{-\sqrt{m}}{7}$$ (+2pts)

specific solution: $u(t) = \frac{1}{4} \cos\left(\frac{7}{\sqrt{m}}t\right) - \frac{\sqrt{m}}{7} \sin\left(\frac{7}{\sqrt{m}}t\right)$ (+2pts)

amplitude $= R = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{-\sqrt{m}}{7}\right)^2} = \sqrt{\frac{1}{16} + \frac{m}{49}} = \frac{1}{2}$ (algebra) $\rightsquigarrow$ $\boxed{m = \frac{147}{16} \text{ kg}}$

≈ 9.19 kg

(b) (3 points) That cake is removed and eaten. Now suppose I want to attach a different cake to the same spring, so that it oscillates exactly once every second. What mass should the cake have? Include units.

$\omega_0 = \sqrt{\frac{k}{m}}$, $\quad T = \frac{2\pi}{\omega_0} = 2\pi \cdot \frac{\sqrt{m}}{7} = 1$ sec

$$\downarrow$$

$$\sqrt{m} = \frac{7}{2\pi} \longrightarrow \boxed{m = \frac{49}{4\pi^2} \text{ kg}}$$

≈ 1.24 kg