# Math 307, Spring 2012

Midterm 2A Wednesday, May 16th

**Name:** Solutions

Student ID #__________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

| | | |
|---|---|---|
| 1 | 28 | |
| 2 | 12 | |
| 3 | 10 | |
| Total | 50 | |

- Please check that your exam contains 3 problems on 6 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

GOOD LUCK!

**1.** (4 points each; 28 points total) For parts (a) through (f), find the general solution for the given differential equation.

(a) $y'' + y' + 4y = 0$.

$r^2 + r + 4 = 0$ characteristic eqn.

roots: $r = \frac{-1 \pm \sqrt{1-4(1)(4)}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{15}}{2}$

(2 pts)

$$\boxed{y(t) = e^{-\frac{1}{2}t}\left(c_1 \cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{15}}{2}t\right)\right)}$$

(2 pts)

(b) $y'' - 4y' + 4y = 0$.

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2$ repeated root

$$\boxed{y(t) = c_1 e^{2t} + c_2 t e^{2t}}$$

(c) $y'' - 4y = 0$.

$r^2 - 4 = 0$

$(r-2)(r+2) = 0$

$r = \pm 2$

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t}}$$

(d) $y'' - 4y' = 0$.

$r^2 - 4r = 0$

$r(r-4) = 0$

$r = 0, 4$

$$\boxed{y(t) = c_1 + c_2 e^{4t}}$$

**(e)** $y'' + 4y = \sin 2t$. $\quad y(t) = y_c(t) + Y(t)$

For $y_c(t)$:

$r^2 + 4 = 0$

$r = \pm 2i$

$y_c(t) = c_1 \cos 2t + c_2 \sin 2t$

(1pt)

for $Y(t)$:

guess $Y = At\cos 2t + Bt\sin 2t$ (1pt)

since $\cos 2t$ and $\sin 2t$ are sol'ns to homogeneous equation

then $Y'(t) = -2At\sin 2t + A\cos 2t + 2Bt\cos 2t + B\sin 2t$

$Y'' = -4Bt\sin 2t - 4At\cos 2t - 4A\sin 2t + 4B\cos 2t$ (1pt)

plug into $y'' + 4y = \sin 2t$, get

$-4A\sin 2t + 4B\cos 2t = \sin 2t$

so $-4A = 1$ and $B = 0$

$A = -\frac{1}{4}$ (1pt)

gen. sol'n:

$$\boxed{y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}t\cos 2t}$$

**(f)** $y'' + 4y = \sin 3t$.

$y_c(t)$ same as in (e)

For $Y(t)$, guess $Y(t) = A\cos 3t + B\sin 3t$ (1pt)

then $Y' = -3A\sin 3t + 3B\cos 3t$

$Y'' = -9A\cos 3t - 9B\sin 3t$ (1pt)

plug into $y'' + 4y = \sin 3t$ and get

$(-9A + 4A)\cos 3t + (-9B + 4B)\sin 3t = \sin 3t$

$-5A = 0$ $\quad\quad$ $-5B = 1$

$A = 0$ $\quad\quad$ $B = -\frac{1}{5}$ (1pt)

general solution: $\boxed{y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{5}\sin 3t}$

(1pt)

(g) Write down what guess you would use for the particular solution to the following differential equation. DO NOT solve the equation.

$$y'' + 4y = t^2 \sin 3t - e^{2t}.$$

$Y(t) = (At^2 + Bt + C)(D\cos 3t + E\sin 3t) + Fe^{2t}$

2. (3 points each; 12 points total) Suppose you are given a damped spring-mass system that satisfies the equation $mu'' + \gamma u' + ku = 0$, with $u(0) = u_0$ and $u'(0) = v_0$. Circle or write TRUE or FALSE.

(a) TRUE or FALSE. When $\gamma > 0$, the quasi-frequency is always less than the natural frequency.

$\dfrac{\sqrt{4mk - \gamma^2}}{2m}$ $\qquad\qquad$ $\omega_0 = \sqrt{\dfrac{k}{m}} = \dfrac{\sqrt{4km}}{2m}$

(b) TRUE or FALSE. For all possible values of $m, \gamma$, and $k$, the solution $u(t)$ is defined for (i.e. its domain of validity includes) all $t \geq 0$.

we know this b/c we found all solutions, and they are defined for all $t$.

(c) TRUE or FALSE. It is possible to have values for $m$ and $k$ such that no $\gamma$ value will result in critical damping.

critical damping is defined as $\gamma$ such that $\gamma^2 = 4mk$.
given any $m$ and $k$, setting $\gamma = \sqrt{4mk}$ will result in critical damping

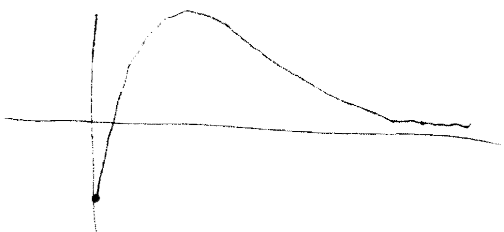
(d) TRUE or FALSE. Suppose $u(0) < 0$. If $\gamma^2 \geq 4mk$, then we must have $u(t) < 0$ for all $t$.

when $\gamma^2 = 4mk$, the general solution is

$u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$, and as discussed in class

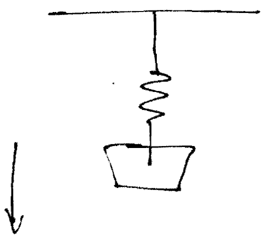
this can cross the $u = 0$ axis zero or one time.





**3.** (10 points total)

**(a)** (7 points) A 63 lbs miniature piano with a 1 lbs cat sleeping on it is suspended by a spring. Together, they stretch the spring 64/29 feet. There is a damper in the spring, that would exert a friction force of 12 lbs on a weight moving at 2 ft/s. The piano is pulled down 1 foot and released. The cat stays on the piano. Find a formula that describes the motion of the piano and cat as a function of time.



assume down is positive

$$\text{weight} = 64\text{ lbs} = mg = m \cdot 32\,\text{ft}/s^2 \longrightarrow m = 2$$

$$F_{static} = k \cdot \text{displacement}$$
$$64\text{ lbs} = k \cdot (64/29) \longrightarrow k = 29$$

$$F_{damping} = \gamma \cdot \text{velocity}$$
$$12\text{ lbs} = \gamma \cdot 2 \longrightarrow \gamma = 6$$

(+2 pts)

initial value problem is: $2u'' + 6u' + 29u = 0$, $u(0) = 1$, $u'(0) = 0$ ("released")

roots: $\dfrac{-6 \pm \sqrt{36 - 4\cdot 2\cdot 29}}{4} = \dfrac{-3}{2} \pm i\dfrac{7}{2}$

so $u(t) = e^{-\frac{3}{2}t}\cdot c_1 \cos\left(\frac{7}{2}t\right) + e^{-\frac{3}{2}t}\cdot c_2 \sin\left(\frac{7}{2}t\right)$ (+1 pt)

use initial conditions to get $c_1$ and $c_2$:

$u(0) = 1\cdot c_1 \cdot 1 + 1\cdot c_2 \cdot 0 = c_1 = 1$ (+1 pt)

$u'(t) = -e^{-\frac{3}{2}t}\sin\left(\frac{7}{2}t\right)\cdot\frac{7}{2} - \frac{3}{2}e^{-\frac{3}{2}t}\cos\left(\frac{7}{2}t\right) + \frac{7}{2}e^{-\frac{3}{2}t}\cos\left(\frac{7}{2}t\right)c_2 - \frac{3}{2}c_2 e^{-\frac{3}{2}t}\sin\left(\frac{7}{2}t\right)$

$u'(0) = 0 - \frac{3}{2} + \frac{7}{2}c_2 - 0 = 0$

$\frac{7}{2}c_2 = \frac{3}{2} \longrightarrow c_2 = \frac{3}{7}$ (+2 pts)

so specific sol'n is

$$\boxed{u(t) = e^{-\frac{3}{2}t}\cos\left(\frac{7}{2}t\right) + \frac{3}{7}e^{-\frac{3}{2}t}\sin\left(\frac{7}{2}t\right)}$$ (+1 pt)

(a) (3 points) Write your answer to part (a) using one shifted cosine function (rather than a sum of sine and cosine functions). If you don't have a calculator, you can leave your answers in exact form, but make sure to indicate which quadrant the shift $\delta$ is in!

$$1\cos\left(\frac{7}{2}t\right) + \frac{3}{7}\sin\left(\frac{7}{2}t\right) = R\cos\left(\frac{7}{2}t - \delta\right) \text{ where}$$

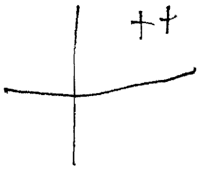
$$R = \sqrt{A^2+B^2} = \sqrt{1+\left(\frac{3}{7}\right)^2} = \sqrt{\frac{49+9}{49}} = \frac{\sqrt{58}}{7}. \quad (+1\text{ pt})$$

and $\tan(\delta) = B/A = 3/7$ so $\delta = \tan^{-1}(3/7)$ (+1 pt)

which quadrant is $\delta$?

$R\cos(\delta) = A = 1 > 0$

$R\sin(\delta) = B = \frac{3}{7} > 0$




in 1st quadrant (+1 pt)

$$\text{so } u(t) = \frac{\sqrt{58}}{7} e^{-\frac{3}{2}t}\cos\left(\frac{7}{2}t - \delta\right)$$

where $\delta = \tan^{-1}\left(\frac{3}{7}\right)$ in 1st quadrant

($\delta \approx 0.405$ radians)