Math 307, Spring 2012

Midterm 1B Wednesday, April 18th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________________

1	10	
2	5	
3	10	
4	10	
5 or 5'	15	
EC	4	
Total	50	

- Please check that your exam contains 5 problems on 7 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

GOOD LUCK!

1. (10 points total) Consider the following initial value problem.

$$ty' + 2y = \frac{\sin(t)}{t}, \quad y(\pi) = 0.$$

(a) (4 pts) Solve for a general solution.

$y' + \left(\frac{2}{t}\right)y = \frac{\sin(t)}{t^2}$ is linear

integrating factor: $\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = e^{2\ln t} = t^2$ (2 pts)

$y(t) = \frac{1}{\mu}\left[\int t^2 \cdot \frac{\sin t}{t^2}\,dt\right] = \frac{1}{t^2}\left[-\cos t + C\right] = \cancel{-\frac{\cos t}{t^2} + t^2}$

$$\boxed{y(t) = \frac{-\cos t}{t^2} + \frac{C}{t^2}}, \; t \neq 0 \quad (2\text{pts})$$

(b) (4 pts) Solve for a specific solution. Include the domain of validity of your solution.

plug in I.C.:

$0 = y(\pi) = \frac{-\cos(\pi) + C}{(\pi)^2} \longrightarrow \boxed{y(t) = \frac{-\cos t}{t^2} - \frac{1}{t^2}}$ (2 pts)

$0 = 1 + C$

$C = -1$

domain of validity is largest interval containing $t_0 = \pi$ on which $\frac{2}{t}$ and $\frac{\sin t}{t^2}$ are continuous, so

$\boxed{0 < t < \infty}$ (2 pts)

i.e. $0 < t$

(c) (2 pts) What is the behavior of the solution as t goes to infinity?

$|\cos(t)| \leq 1$, so as $t \to \infty$, $y \to 0$ (2 pts)

2. (5 points total)

$y(x) =$

(a) (3 pts) Write down a differential equation whose solution is $y(t) = x^2 - 1$.

$y' = 2x$

$y' + y = x^2 + 2x - 1$

etc. many options

(b) (2 pts) Write down a differential equation that can't be solved using any of the techniques we've learned so far this quarter. Explain why not.

- anything with a 2nd derivative, e.g. $y'' - y = 0$ (not first-order so we haven't learned yet)

- anything of the form $M + Ny' = 0$ that isn't exact, and $\frac{M_y - N_x}{N}$ is not a function of x.

  e.g. $x^2 y + x y^2 y' = 0$

3. (10 points total) After an accidental leak, a laboratory room contains 1000 liters of air with a 10% concentration of chlorine gas (i.e. 10 liters of chlorine in every 100 liters of air). When the scientist turns on the exhaust fan, the (well-mixed) contaminated air is blown out of the room at a rate of 50 liters per minute. Fresh air enters the room from a vent at the same rate.

(a) (3 pts) Write down an initial value problem for $Q(t)$, the amount of chlorine in the room.

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0 - \frac{Q(t)}{1000} \times 50$$

$$= \frac{-50}{1000} Q$$

initial condition: 10% of $1000L = 100$

$$\boxed{\frac{dQ}{dt} = -\frac{Q}{20} \qquad Q(0) = 100}$$

(3 pts)

(2 pts)

(b) (4 pts) Find a formula for $Q(t)$.

use separation of variables (or could use integrating factor)

$$\frac{dQ}{Q} = \frac{-1}{20} dt$$

$$\ln|Q| = \frac{-t}{20} + K$$

$$Q(t) = Ce^{-\frac{t}{20}}$$

(3 pts)

plug in i.c.: $Q(0) = 100 = Ce^0 = C$

so $C = 100$

(1 pt)

$$\boxed{Q(t) = 100e^{-\frac{t}{20}}}$$

(c) (3 pts) How long does it take for the chlorine concentration to reach 1%?

1% means 10 L in 1000L room

so, what t gives $Q(t) = 10$? (2 pts)

$$100e^{-\frac{t}{20}} = 10$$

$$e^{-\frac{t}{20}} = 1/10$$

$$-\frac{t}{20} = \ln(1/10)$$

$$t = -20\ln\left(\frac{1}{10}\right) = \boxed{20\ln(10) \approx 46.05 \text{ minutes}}$$ (1 pt)

4. (10 points total) Solve the following initial value problem. Give your solution in explicit form, and include the domain of validity.

$$y' = \frac{2-2x}{y}, \qquad y\left(\frac{-1}{2}\right) = \sqrt{\frac{7}{2}}.$$

separable.

$y\,dy = (2-2x)\,dx$

integrate...

$\frac{y^2}{2} = 2x - x^2 + C$

(3pts)

plug in initial conditions:

$\frac{1}{2}\left(\sqrt{\frac{7}{2}}\right)^2 = 2\left(\frac{-1}{2}\right) - \left(\frac{-1}{2}\right)^2 + C$

$\frac{7}{4} = -1 - \frac{1}{4} + C$

$3 = C$

(3pts)

$\frac{y^2}{2} = 2x - x^2 + 3$

needs to go through $\left(\frac{-1}{2}, \sqrt{\frac{7}{2}}\right)$ so use positive solution

$y^2 = 4x - 2x^2 + 6$

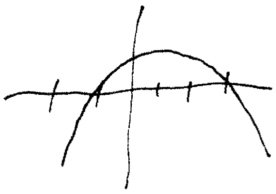
$y = \pm\sqrt{4x-2x^2+6}$

$\boxed{y = \sqrt{4x-2x^2+6}}$ (2pts)

- can't have $y=0$ (from original D.E.)
- can't have $4x-2x^2+6 \leq 0$. when does this happen?

use quadratic formula on $-2x^2+4x+6$

$x = \frac{-4 \pm \sqrt{16+4\cdot 2\cdot 6}}{-4} = \frac{-4 \pm 8}{-4}$ so $x = 3$ or -1



interval of definition is

$\boxed{-1 < x < 3}$ (2pts)

IMPORTANT: Choose ONE of the following problems, 5 or 5', and solve it completely. Only one will be graded. Circle one of the following numbers to let me know which you want me to grade:

5 5'

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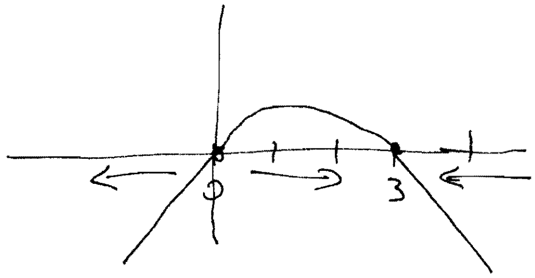
5. (15 points total) Consider the following differential equation. for all y in the range $-\infty < y < \infty$.

$$\frac{dy}{dt} \quad \frac{dy}{dx} = -(e^y - 1)(y - 3).$$

(a) (3 pts) Sketch (*very* roughly) a graph of $f(y)$ vs. y.



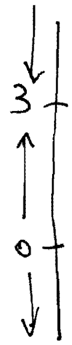
For large y, both terms positive
For large negative y, both negative
between 0 and 3, $e^y - 1 > 0$ but $y - 3 < 0$
(or just plug in a few values)

(b) (3 pts) Identify the equilibrium solutions.

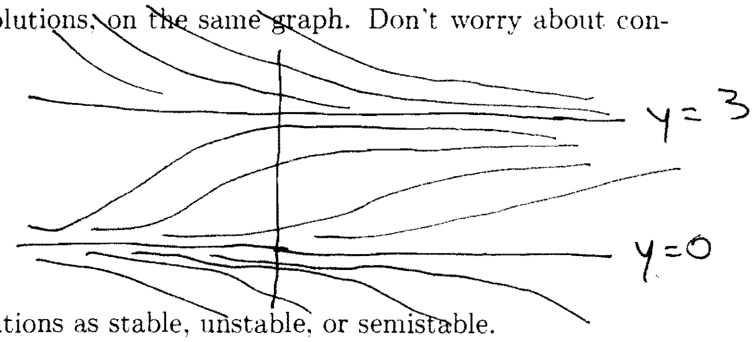
$f(y) = -(e^y - 1)(y - 3) = 0$

$\boxed{y = 0}$ or $\boxed{y = 3}$

(c) (3 pts) Make a phase line.



(d) (3 pts) Sketch several possible solutions on the same graph. Don't worry about concavity.



(e) (3 pts) Label the equilibrium solutions as stable, unstable, or semistable.

$y = 0$ unstable
$y = 3$ stable

5'. (15 points total) Consider the following differential equation.

$$\underbrace{(3y + 2x)}_{M} + \underbrace{x}_{N}y' = 0.$$

(a) (3 pts) Show that this equation is not exact.

$M_y = 3$

$N_x = 1$

not equal

(b) (4 pts) Find an integrating factor of the form $\mu(x)$.

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) = \exp\left(\int \frac{2}{x}\,dx\right) = e^{2\ln x} = x^2$$

$$\boxed{\mu(x) = x^2}$$

(c) (3 pts) Show that your integrating factor makes the equation exact.

$$\underbrace{(3x^2y + 2x^3)}_{\tilde{M}} + \underbrace{(x^3)}_{\tilde{N}}y' = 0$$

$\tilde{M}_y = 3x^2$

$\tilde{N}_x = 3x^2$

same!

(d) (5 pts) Using this integrating factor, solve the original differential equation.

integrate $\Psi_y = \tilde{N} = x^3$

$\Psi(x,y) = x^3y + h(x)$

take partial w.r.t. x:

$\Psi_x = 3x^2y + h'(x) \longleftarrow$ this is $\tilde{M} = \cancel{x}\, 3x^2y + 2x^3$

so $h'(x) = 2x^3$

$h(x) = \frac{1}{2}x^4$

$$\boxed{\Psi(x,y) = x^3y + \frac{1}{2}x^4 = C}$$

EXTRA CREDIT. (4 points total) Solve the other one of 5 and 5'.