Math 307, Spring 2012

Midterm 1A   Wednesday, April 18th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________________

1	10	
2	10	
3	10	
4	5	
5 or 5'	15	
EC	4	
Total	50	

- Please check that your exam contains 5 problems on 7 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

GOOD LUCK!

1. (10 points total) After an accidental leak, a laboratory room contains 1000 liters of air with a 10% concentration of chlorine gas (i.e. 10 liters of chlorine in every 100 liters of air). When the scientist turns on the exhaust fan, the (well-mixed) contaminated air is blown out of the room at a rate of 100 liters per minute. Fresh air enters the room from a vent at the same rate.

(a) (3 pts) Write down an initial value problem for $Q(t)$, the amount of chlorine in the room.

$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$

$0 \quad - \quad \frac{Q(t)}{1000} \times 100$

$$\boxed{\frac{dQ}{dt} = -\frac{Q}{10} \quad (2\text{ pts}) \qquad Q(0) = 100 \quad (1\text{ pt})}$$

initial condition: 10% of 1000L = 100

(b) (4 pts) Find a formula for $Q(t)$.

use separation of variables (could use integrating factor too)

$\frac{dQ}{Q} = -\frac{1}{10}dt$

$\ln|Q| = \frac{-t}{10} + K$

(3pts) $Q(t) = Ce^{-t/10}$

plug in i.c.: $Q(0) = C \cdot 1 = 100$ so $C = 100$ (1pt)

$$\boxed{Q(t) = 100e^{-t/10}}$$

(c) (3 pts) How long does it take for the chlorine concentration to reach 1%?

1% means 10L chlorine in 1000L room

so want t such that $Q(t) = 10$. (2pts)

$100e^{-t/10} = 10$

$e^{-t/10} = 1/10$

$-\frac{t}{10} = \ln\left(\frac{1}{10}\right)$

$\boxed{t = -10 \cdot \ln(1/10)} \approx \boxed{23.03 \text{ minutes}}$ (1pt)

2. (10 points total) Consider the following initial value problem.

$$ty' + 2y = \frac{\cos(t)}{t}, \qquad y\left(\frac{-\pi}{2}\right) = 0.$$

(a) (4 pts) Solve for a general solution.

$y' + \left(\frac{2}{t}\right) y = \frac{\cos(t)}{t^2}$ is linear

integrating factor: $\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = e^{2\ln t} = t^2$ (2 pts)

$y(t) = \frac{1}{\mu}\left[\int t^2 \cdot \left(\frac{\cos t}{t^2}\right) dt\right] = \frac{1}{t^2}\left[\sin t + C\right]$ so $\boxed{y(t) = \frac{\sin(t) + C}{t^2}, \; t \neq 0}$

(2 pts)

(b) (4 pts) Solve for a specific solution. Include the domain of validity of your solution.

plug in I.C.:

$y\left(\frac{-\pi}{2}\right) = \frac{-1 + C}{\left(\frac{-\pi}{2}\right)^2} = 0$

so $C = 1$

(2 pts) $\boxed{y(t) = \frac{\sin(t) + 1}{t^2}}$

d.o.v.: largest interval containing $t_0 = \frac{-\pi}{2}$ on which $\frac{2}{t}$ and $\frac{\cos t}{t^2}$ are continuous:

$-\infty < t < 0$ (2 pts)

i.e. $t < 0$

(c) (2 pts) What is the behavior of the solution as t goes to infinity?

$|\sin(t)| \leq 1$

(2 pts) as $t \to \infty$, $y(t) \to 0$

3. (10 points total) Solve the following initial value problem. Give your solution in explicit form, and include the domain of validity.

$$y' = \frac{1-2x}{y}, \quad y(1) = -2.$$

separable.

$y\,dy = (1-2x)\,dx$

integrate...

$\frac{y^2}{2} = x - x^2 + C$

(3 pts)

plug in initial condition:

$\frac{(-2)^2}{2} = (1) - (1)^2 + C \longrightarrow C = 2$

$\frac{y^2}{2} = x - x^2 + 2$

$y^2 = 2x - 2x^2 + 4$ (3 pts)

$y = \pm\sqrt{2x - 2x^2 + 4}$

needs to go through $\underset{y(1)=-2}{(1,-2)}$ so use negative solution

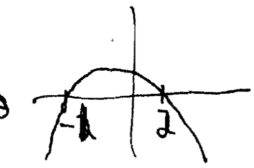
$$\boxed{y(x) = -\sqrt{2x - 2x^2 + 4}}$$ (2 pts)

- can't have $y = 0$ (from original D.E.)
- can't have $2x - 2x^2 + 4 \leq 0$. When does this happen?

quadratic formula for $-2x^2 + 2x + 4$: $x = \frac{-2 \pm \sqrt{4 + 4(2)(4)}}{-4}$

$x = \frac{-2 \pm 6}{-4}$

$x = -1$ or $+2$



so interval of definition is

$$\boxed{-1 < x < 2}$$ (2 pts)

4. (5 points total)

(a) (3 pts) Write down a differential equation whose solution is ~~$y(t)$~~ $y(x) = x^2 + 1$.

$y' = 2x$

$y' + y = x^2 + 2x + 1$

etc. many options

(b) (2 pts) Write down a differential equation that can't be solved using any of the techniques we've learned so far this quarter. Explain why not.

- anything that's not first order. e.g. $y'' - y = 0$
- anything of the form $M(x,y) + N(x,y)y' = 0$ that isn't exact, and such that $\frac{M_y - N_x}{N}$ is not a function of x only.

e.g. $(x^2y) + (xy^2)y' = 0$.

IMPORTANT: Choose ONE of the following problems, 5 or 5', and solve it completely. Only one will be graded. Circle one of the following numbers to let me know which you want me to grade:

5 5'

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5. (15 points total) Consider the following differential equation.

$$\underbrace{(2y+3x)}_{M} + \underbrace{x}_{N}y' = 0.$$

(a) (3 pts) Show that this equation is not exact.

$M_y = 2$ not equal

$N_x = 1$

(b) (4 pts) Find an integrating factor of the form $\mu(x)$.

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) = \exp\left(\int \frac{1}{x}\,dx\right) = e^{\ln|x|} = x$$

$$\boxed{\mu(x) = x}$$

(c) (3 pts) Show that your integrating factor makes the equation exact.

$$\underbrace{(2xy + 3x^2)}_{\tilde{M}} + \underbrace{(x^2)}_{\tilde{N}}y' = 0$$

$$\tilde{M}_y = 2x \xleftrightarrow[\text{SAME!}]{} \tilde{N}_x = 2x$$

(d) (5 pts) Using this integrating factor, solve the original differential equation.

integrate $\Psi_x = \tilde{M} = 2xy + 3x^2$ w.r.t. x:

$\Psi(x,y) = x^2 y + x^3 + h(y)$

take partial w.r.t. y:

$\Psi_y = x^2 + h'(y) \leftarrow$ this is $\tilde{N} = x^2$

so $h'(y) = 0$

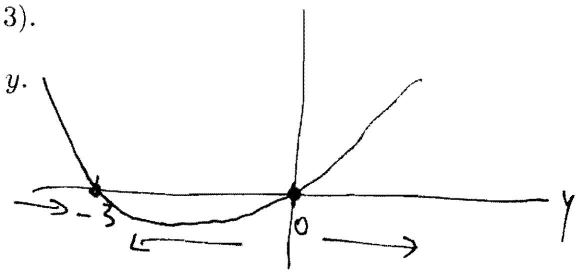
$h(y) =$ constant. choose $h(y) = 0$.

$$\boxed{x^2 y + x^3 = C}$$

5'. (15 points total) Consider the following differential equation, for all y in the range $-\infty < y < \infty$.

$$\frac{dy}{dt}\ \cancel{\frac{dy}{dx}} = (e^{2y} - 1)(y + 3).$$

(a) (3 pts) Sketch (*very* roughly) a graph of $f(y)$ vs. y.



for large y, both terms positive

for large negative y, both terms negative

between -3 and 0, $e^{2y}-1<0$ and $y+3>0$

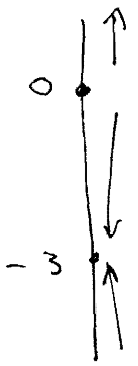
(or just plug in a few values)

(b) (3 pts) Identify the equilibrium solutions.

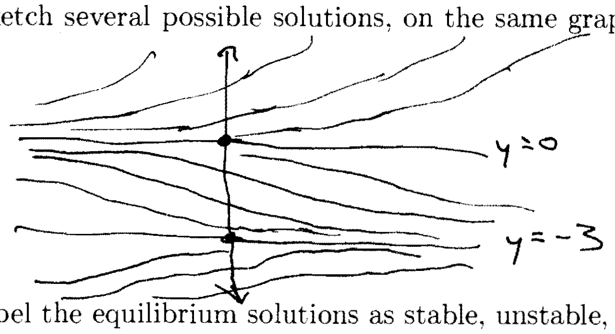
$$f(y) = 0 = (e^{2y}-1)(y+3)$$

$\boxed{y=0}$ or $\boxed{y=-3}$

(c) (3 pts) Make a phase line.



(d) (3 pts) Sketch several possible solutions, on the same graph. Don't worry about concavity.



(e) (3 pts) Label the equilibrium solutions as stable, unstable, or semistable.

$y=0$ is unstable

$y=-3$ is stable

EXTRA CREDIT. (4 points total) Solve the other one of 5 and 5'.