Math 324, Autumn 2011

Midterm 2B Wednesday, November 16th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________________

1	12	
2	11	
3	12	
4	15	
EC	5	
Total	50	

- Please check that your exam contains 4 problems on 6 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes. You don't need to turn it in.

GOOD LUCK!

1. (5+7=12 points total)

(a) Let $f(x, y, z) = xe^y \cos(z)$. What is ∇f at $(2, 0, 0)$?

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle e^y \cos z,\ x e^y \cos z,\ -x e^y \sin z \rangle \qquad (3\text{pts})$$

$$\nabla f(2,0,0) = \langle e^0 \cos(0), (2)(e^0)\cos(0), -2e^0 \sin(0) \rangle \qquad (1\text{pt})$$

$$= \boxed{\langle 1, 2, 0 \rangle} \qquad (1\text{pt})$$

(b) Let $g(x, y) = 2x^2 + y^2$. Find two points in the plane where the change of g in the direction of $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$ is zero.

want (x_0, y_0) so that $D_{\vec{u}}\, g(x_0, y_0) = 0$

$$\mathrel{\rotatebox{90}{=}}\ \nabla g \cdot \vec{u}$$

$$\nabla g = \langle 4x, 2y \rangle$$

$$\nabla g \cdot \vec{u} = \langle 4x, 2y \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle = 2x\sqrt{2} - y\sqrt{2} = 0 \qquad (5\text{pts})$$

$$2x - y = 0$$

$$y = 2x$$

so any two points on $y = 2x$ will work.

e.g. $\boxed{\begin{matrix}(0,0)\\(1,2)\end{matrix}}$ $\qquad (2\text{pts})$

$(-1,-2)$ etc...

2. (7+4=11 points total) Consider the line integral

$$I = \int_C xy^2z\,ds,$$

where C is the line segment from $(-1, 0, 5)$ to $(1, 4, 6)$.

(a) Put I in the form $\int_a^b p(t)dt$, for some function $p(t)$ and some constants a and b.

parametrize C as $\vec{r}(t) = \langle -1, 0, 5\rangle + t\langle 1-(-1), 4-0, 6-5\rangle$ $\quad 0 \le t \le 1$

$$\vec{r}(t) = \langle 2t-1, 4t, t+5\rangle \qquad (3\text{ pts})$$

$$\vec{r}\,'(t) = \langle 2, 4, 1\rangle \quad \text{so} \quad |\vec{r}\,'(t)| = \sqrt{(2)^2+(4)^2+(1)^2} = \sqrt{21} \qquad (3\text{ pts})$$

$$I = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \boxed{\int_0^1 \underbrace{(2t-1)}_{x}\underbrace{(4t)^2}_{y^2}\underbrace{(t+5)}_{z}\cdot\sqrt{21}\,dt} \qquad (1\text{ pt})$$

(b) Evaluate I.

$$(2t-1)(16t^2)(t+5)\cdot\sqrt{21} = \sqrt{21}\cdot(2t^2+9t-5)(16t^2) = \sqrt{21}\cdot(32t^4+144t^3-80t^2) \qquad (2\text{ pts})$$

$$\text{so } I = \int_0^1 \sqrt{21}\cdot(32t^4+144t^3-80t^2)\,dt = \sqrt{21}\left[\frac{32}{5}t^5 + \frac{144}{4}t^4 - \frac{80}{3}t^3\right]_0^1$$

$$= \sqrt{21}\left(\frac{32}{5} + 36 - \frac{80}{3}\right) = \boxed{\frac{236}{15}\sqrt{21}} \qquad (2\text{ pts})$$

3. (12 points total) Consider the vector field $\vec{F}(x, y, z) = \langle z,\, y^2,\, -x\rangle$ and the curve C parametrized as $\vec{r}(t) = \langle \sin t,\, -t,\, \cos t\rangle$, where $0 \le t \le 2\pi$. Calculate the line integral of $\vec{F}$ along C,

$$\int_C \vec{F} \cdot d\vec{r}. = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\vec{r}\,'(t) = \langle \cos t, -1, -\sin t\rangle \qquad (3pts)$$

$$F(\vec{r}(t)) = \langle \cos t,\ (-t)^2,\ -(\sin t)\rangle \qquad (3pts)$$

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = \cos^2 t - t^2 + \sin^2 t = 1 - t^2 \qquad (3pts)$$

$$\text{so } \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (1 - t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^{2\pi} = 2\pi - \frac{(2\pi)^3}{3} = \boxed{2\pi - \frac{8\pi^3}{3}} \qquad (3pts)$$

4. (5+5+5=15 points total) Let C be the curve from $(2,4)$ to $(3,9)$ along the function $x = \sqrt{y}$. Consider the vector field

$$\vec{F}(x,y) = \langle y - x,\ x + y \rangle .$$

(a) Show that $\vec{F}$ is conservative.

$\vec{F}$ is defined on all the plane (which has no holes and is open)

need $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = 1$. equal, so $\vec{F}$ is conservative. (5pts)

(b) Find a function f such that $\nabla f = \vec{F}$.

want f so $f_x = y - x$ $\longrightarrow$ $f = xy - \frac{x^2}{2} + h(y)$ for some $h(y)$ (2pts)

and $f_y = x + y$ $\longrightarrow$ $f = xy + \frac{y^2}{2} + g(x)$ for some $g(x)$ (2pts)

$\boxed{f(x,y) = xy - \frac{x^2}{2} + \frac{y^2}{2}}$ works. could also add any constant. (1pt)

(c) Calculate $\int_C \vec{F} \cdot d\vec{r}$.

(3pts) (2pts)

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = f(3,9) - f(2,4) = \boxed{49}$

$f(3,9) = 3 \cdot 9 - \frac{3^2}{2} + \frac{9^2}{2} = 27 + 36 = 63$

$f(2,4) = 2 \cdot \cancel{4} - \frac{2^2}{2} + \frac{4^2}{2} = \cancel{8}\ 8 - 2 + 8 = 14$

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EXTRA CREDIT ONE. (3 points total) Let C be the circle with radius three, parametrized counter-clockwise. Use Green's Theorem to evaluate

$$\int_C x^2y\,dx - xy^2\,dy.$$

$P = x^2y$ $\quad \dfrac{\partial P}{\partial y} = x^2$

$Q = -xy^2$ $\quad \dfrac{\partial Q}{\partial x} = -y^2$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left(-y^2 - x^2\right) dA = -\iint_D (x^2+y^2)\, dA$$

$$= -\int_0^{2\pi}\int_0^3 (r^2)\, r\,dr\,d\theta = -\int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^3 = -\int_0^{2\pi} \frac{81}{4}\,d\theta = \boxed{-\frac{81\pi}{2}}$$

(all or nothing)

EXTRA CREDIT TWO. (2 points total) Let $z = e^r \cos\theta$, and $r = st$, and $\theta = \sqrt{s^2+t^2}$. Find $\frac{\partial z}{\partial t}$ when $s = t = 1$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = \left(e^r\cos\theta\right)\cdot(s) + \left(-e^r\sin\theta\right)\cdot\frac{1}{\cancel{2}}\cdot(s^2+t^2)^{-1/2}\cdot(\cancel{2}t)$$

when $s=1$, $t=1$ then $r = 1$ and $\theta = \sqrt{1+1} = \sqrt{2}$

$$\text{so } \frac{\partial z}{\partial t}(1,1) = \left(e^1\cos\sqrt{2}\right)(1) - \frac{e^1\sin\sqrt{2}}{\sqrt{1+1}}\cdot(1) = \boxed{e\cos\sqrt{2} - \frac{e}{\sqrt{2}}\sin\sqrt{2}}$$

(all or nothing)