Math 324, Autumn 2011

Midterm 2A   Wednesday, November 16th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

1	12	
2	11	
3	12	
4	15	
EC	5	
Total	50	

- Please check that your exam contains 4 problems on 6 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes. You don't need to turn it in.

GOOD LUCK!

1. (5+7=12 points total)

(a) Let $f(x, y, z) = e^x y \sin(z)$. What is ∇f at $(0, -1, 0)$?

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle e^x y \sin z,\ e^x \sin z,\ e^x y \cos z \rangle \qquad (3pts)$$

$$\nabla f(0,-1,0) = \langle e^0 \cdot (-1) \cdot \sin(0),\ e^0 \cdot \sin(0),\ e^0 \cdot (-1) \cdot \cos(0) \rangle \qquad (1pt)$$

$$= \boxed{\langle 0, 0, -1 \rangle} \qquad (1pt)$$

(b) Let $g(x, y) = x^2 + 3y^2$. Find two points in the plane where the change of g in the direction of $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$ is zero.

want (x_0, y_0) so that $D_{\vec{u}}\, g(x_0, y_0) = 0$

$$= \nabla g \cdot \vec{u}$$

$$\nabla g = \langle g_x, g_y \rangle = \langle 2x, 6y \rangle$$

$$\nabla g \cdot \vec{u} = \langle 2x, 6y \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle = x\sqrt{2} - 3y\sqrt{2} = 0 \qquad (5pts)$$

$$x - 3y = 0 \rightarrow y = \tfrac{1}{3}x$$

so any two points on $y = \frac{1}{3}x$ will work.

e.g. $\boxed{\begin{matrix}(0,0)\\(3,1)\end{matrix}}$ $\qquad (2pts)$

$(-3,-1)$ etc...

2. (7+4=11 points total) Consider the line integral

$$I = \int_C xyz^2\,ds,$$

where C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

(a) Put I in the form $\int_a^b p(t)dt$, for some function $p(t)$ and some constants a and b.

parametrize C as $\vec{r}(t) = \langle -1, 5, 0\rangle + t\langle 1-(-1), 6-5, 4-0\rangle \quad 0 \le t \le 1$

$$\vec{r}(t) = \langle \overset{x}{2t-1}, \overset{y}{t+5}, \overset{z}{4t}\rangle \qquad (3pts)$$

$$\vec{r}\,'(t) = \langle 2, 1, 4\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(2)^2+(1)^2+(4)^2} = \sqrt{21} \qquad (3pts)$$

$$\text{so } I = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \boxed{\int_0^1 (2t-1)(t+5)(4t)^2 \cdot \sqrt{21}\,dt} \qquad (1pt)$$

$x \quad y \quad z^2$

(b) Evaluate I.

$$(2t-1)(t+5)(16t^2)\sqrt{21} = \sqrt{21}\cdot(2t^2+9t-5)(16t^2) = \sqrt{21}\,(32t^4 + 144t^3 - 80t^2)$$

$$\text{so } I = \sqrt{21}\int_0^1 32t^4 + 144t^3 - 80t^2\,dt \qquad (2pts)$$

$$= \sqrt{21}\left[\frac{32t^5}{5} + \frac{144t^4}{4} - \frac{80t^3}{3}\right]_0^1 = \sqrt{21}\left[\frac{32}{5} + 36 \quad \frac{80}{3}\right] = \boxed{\frac{236}{15}\sqrt{21}}$$

(2pts)

3. (12 points total) Consider the vector field $\vec{F}(x, y, z) = \langle x,\ y^2,\ z\rangle$ and the curve C parametrized as $\vec{r}(t) = \langle \sin t,\ -t,\ \cos t\rangle$, where $0 \le t \le 2\pi$. Calculate the line integral of $\vec{F}$ along C,

x y z

$$\int_C \vec{F} \cdot d\vec{r}. = \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\vec{r}\,'(t) = \langle \cos t, -1, -\sin t \rangle \qquad (3\text{pts})$$

$$F(\vec{r}(t)) = \langle \sin t, (-t)^2, \cos t \rangle \qquad (3\text{pts})$$

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = \cancel{\sin t \cos t} - t^2 - \cancel{\sin t \cos t} \qquad (3\text{pts})$$

so $$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} (-t^2)\, dt = -\frac{t^3}{3}\Big]_0^{2\pi} = -\frac{(2\pi)^3}{3} = \boxed{-\frac{8\pi^3}{3}} \qquad (3\text{pts})$$

4. (5+5+5=15 points total) Let C be the curve from $(2,4)$ to $(3,9)$ along the function $x = \sqrt{y}$. Consider the vector field

$$\vec{F}(x,y) = \left\langle \frac{y^3 - 3x}{3},\ xy^2 + y \right\rangle.$$

(a) Show that $\vec{F}$ is conservative.

$\vec{F}$ is defined on all the plane (which has no holes and is open) (5 pts)

need $\frac{\partial Q}{\partial x} \; \frac{\partial P}{\partial y}$. $\frac{\partial Q}{\partial x} = y^2$ and $\frac{\partial P}{\partial y} = \frac{3y^2}{3} = y^2$. equal, so $\vec{F}$ is conservative.

(b) Find a function f such that $\nabla f = \vec{F}$.

want f so $f_x = \frac{y^3}{3} - x \rightsquigarrow f = \frac{xy^3}{3} - \frac{x^2}{2} + h(y)$ for some $h(y)$. (2 pts)

and $f_y = xy^2 + y \longrightarrow f = \frac{xy^3}{3} + \frac{y^2}{2} + g(x)$ for some $g(x)$. (2 pts)

$\boxed{f(x,y) = \frac{xy^3}{3} + \frac{y^2}{2} - \frac{x^2}{2}}$ works. could also add any constant. (1 pt)

(c) Calculate $\int_C \vec{F} \cdot d\vec{r}$. (3 pts) (2 pts)

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = f(3,9) - f(2,4) = \boxed{\frac{2149}{3}}$

$f(3,9) = \frac{3 \cdot 9^3}{3} + \frac{9^2 - 3^2}{2} = 729 + 36 = 765 = \frac{2295}{3}$

$f(2,4) = \frac{2 \cdot 4^3}{3} + \frac{4^2 - 2^2}{2} = \frac{128}{3} + 6 = \frac{146}{3}$

EXTRA CREDIT ONE. (3 points total) Let C be the circle with radius two, parametrized counter-clockwise. Use Green's Theorem to evaluate

$$\int_C x^2y\,dx - xy^2\,dy.$$

$P = x^2y$ $\quad \frac{\partial P}{\partial y} = x^2$

$Q = -xy^2$ $\quad \frac{\partial Q}{\partial x} = -y^2$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (-y^2 - x^2)\,dA = -\iint_D (x^2+y^2)\,dA$$

$$= -\int_0^{2\pi}\int_0^{2} (r^2)\,r\,dr\,d\theta = -\int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = -\int_0^{2\pi} 4\,d\theta = \boxed{-8\pi}$$

(all or nothing)

EXTRA CREDIT TWO. (2 points total) Let $z = e^r \cos\theta$, and $r = st$, and $\theta = \sqrt{s^2+t^2}$. Find $\frac{\partial z}{\partial s}$ when $s = t = 1$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = (e^r\cos\theta)(t) + (-e^r\sin\theta)\cdot\frac{1}{2}(s^2+t^2)^{-1/2}\cdot 2s$$

when $s=1$, $t=1$, then $r=1$ and $\theta = \sqrt{1+1} = \sqrt{2}$

so $$\frac{\partial z}{\partial s}(1,1) = \left(e^1\cos(\sqrt{2})\right)\cdot(1) - \frac{\left(e^1\sin(\sqrt{2})\right)}{\sqrt{1+1}}\cdot(1)$$

$$= \boxed{e\cos\sqrt{2} - \frac{e\sin\sqrt{2}}{\sqrt{2}}}$$

(all or nothing)