Math 324, Autumn 2011

Midterm 1A Wednesday, October 19th

Name: Solutions

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________________

1	10	
2	5	
3	5	
4	3	
5	8	
6	9	
7	10	
EC	4	
Total	50	

- Please check that your exam contains 7 problems on 8 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

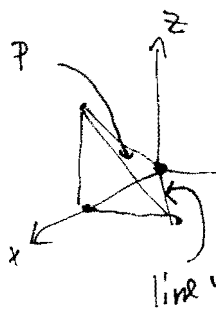
GOOD LUCK!

1. (6+4=10 points total)

(a) Let E be the solid tetrahedron with vertices

$$(0,0,0),(1,0,0),(1,1,0), \text{ and } (1,0,1).$$

Set up **but do not evaluate** the integral of $f(x,y,z)=x^2yz$ over E.



equation of plane P: $ax+by+cz=d$
through $(0,0,0)$, $(1,1,0)$, $(1,0,1)$.

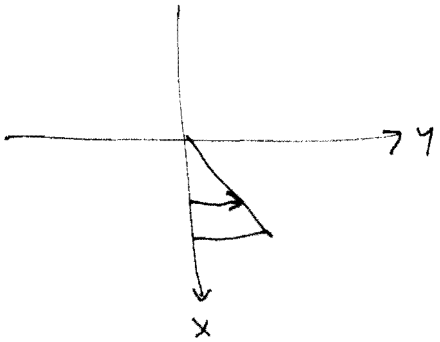
$d=0$ $\quad a+b=0$ $\quad a+c=0$

pick $a=1$
then $b=-1$
$c=-1$
$d=0$.

P is: $x-y-z=0$

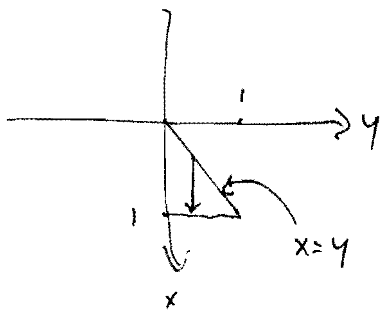
$x-y=z$

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=x-y} x^2yz\,dz\,dy\,dx$$



(b) Set up a different integral of the same function f, over the same tetrahedron E. The order of integration (e.g. $dx\,dy\,dz$) must be different than in the previous problem.

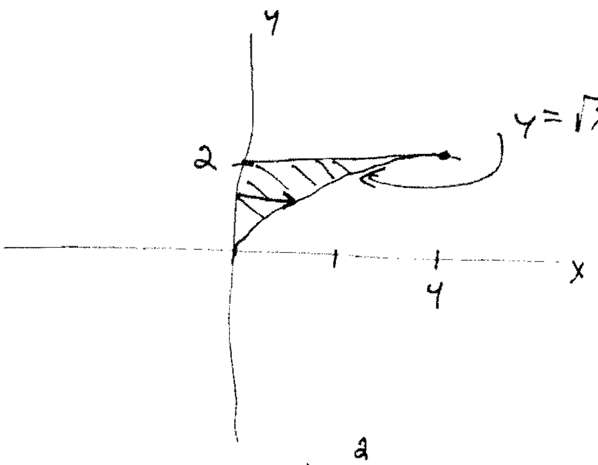
$$\int_{y=0}^{y=1}\int_{x=y}^{x=1}\int_{z=0}^{z=x-y} x^2yz\,dz\,dx\,dy$$



of course, 4 other ways

2. (5 points total) Calculate the iterated integral by reversing the order of integration.

$$I = \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx. = \int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \frac{1}{y^3+1}\, dx\, dy$$



$y = \sqrt{x} \leftrightarrow x = y^2$

inside: $\displaystyle\int_{x=0}^{x=y^2} \frac{1}{y^3+1}\, dx = \frac{y^2 - 0}{y^3+1} = \frac{y^2}{y^3+1}$

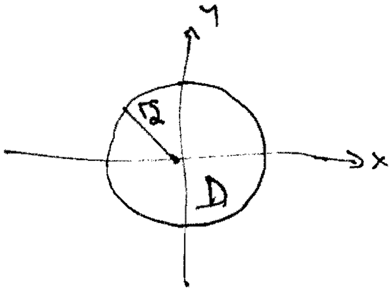
$$I = \int_{y=0}^{y=2} \frac{y^2}{y^3+1}\, dy \qquad u = y^3+1, \quad \frac{du}{3} = \frac{\cancel{3}y^2 dy}{\cancel{3}} \qquad \frac{1}{3}\int \frac{1}{u}\, du = \frac{1}{3}\ln|u|$$

$$= \frac{1}{3}\ln(y^3+1)\Big]_0^2 = \frac{1}{3}\ln(9) - \underbrace{\frac{1}{3}\cancel{\ln(1)}}_{0}$$

$$= \boxed{\frac{1}{3}\ln(9)}$$

3. (5 points total) Evaluate the following integral. convert to polar.

$$I = \iint_D \underbrace{\cos(x^2 + y^2)}_{\cos(r^2)} \, \underbrace{dA}_{r\,dr\,d\theta}, \text{ where } D = \{(x, y) \mid x^2 + y^2 \le 2\}.$$



$$D = \left\{ (r,\theta) : \begin{array}{l} 0 \le r \le \sqrt{2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$I = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \cos(r^2)\, r\,dr\,d\theta = \underbrace{\left[\int_0^{2\pi} 1 \cdot d\theta \right]}_{2\pi} \cdot \left[\int_0^{\sqrt{2}} \cos(r^2)\, r\,dr \right]$$

use $u = r^2$

$$\frac{du}{2} = \frac{\cancel{2}r\,dr}{\cancel{2}}$$

$$\frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u)$$

so $\cancel{\int_0^{\sqrt{2}}}\ \left[\frac{1}{\cancel{2}} \sin(r^2) \right]_{r=0}^{r=\sqrt{2}}$

$$= \frac{1}{2}\left(\sin(2) - \sin(0) \right) = \frac{1}{2}\sin(2)$$

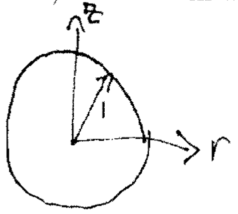
so $I = 2\pi \cdot \frac{1}{2}\sin(2)$

$$\boxed{= \pi \sin(2)}$$

4. (1+1+1=3 points total) Describe in words the surface whose equation is given. Attempt a sketch if you want.

(a) $r^2 + z^2 = 1$.

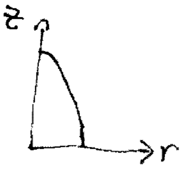
cylindrical, no θ.

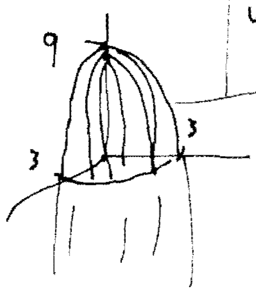


sphere centered at origin with radius 1.

(b) $z = 9 - r^2$.

cylindrical. no θ.





upside down parabolid

(c) $\rho = 2\cos\theta\ \sin\phi$.

spherical.

$\rho^2 = 2\rho\cos\theta\sin\phi$

$x^2+y^2+z^2 = 2x \longrightarrow x^2-2x+1+y^2+z^2 = 1$

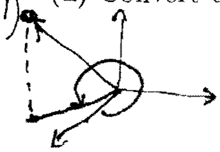
$(x-1)^2+(y^2)+(z^2) = 1$

sphere w/radius 1, centered at (1,0,0).

5. (2+2+2+2=8 points total)

(a) Convert the point $(x, y, z) = (1, -1, 1)$ from rectangular to cylindrical coordinates.

(1,-1,1)



$z = 1$

$\theta = \frac{7\pi}{2}$

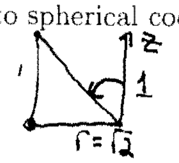
$r = \sqrt{2}$

$\left(\sqrt{2}, \frac{7\pi}{4}, 1\right)$

(b) Convert the same point $(x, y, z) = (1, -1, 1)$ to spherical coordinates.

$\theta = \frac{7\pi}{2}$

$\rho = \sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{3}$

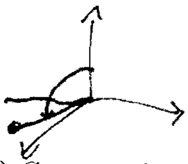


$\tan\phi = \sqrt{2}$

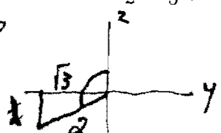
$\phi = \arctan\sqrt{2}$

$\left(\sqrt{3}, \frac{7\pi}{4}, \arctan\sqrt{2}\right)$

(c) Convert the point $(\rho, \theta, \phi) = (2, \frac{3\pi}{2}, \frac{2\pi}{3})$ from spherical to rectangular coordinates.



$x = 0$



$y = -\sqrt{3}$

$z = -1$

$(0, -\sqrt{3}, -1)$

(d) Convert the same point $(\rho, \theta, \phi) = (2, \frac{3\pi}{2}, \frac{2\pi}{3})$ to cylindrical coordinates.

$z = -1$

$\theta = \frac{3\pi}{2}$

$r = \sqrt{3}$

$\left(\sqrt{3}, \frac{3\pi}{2}, -1\right)$

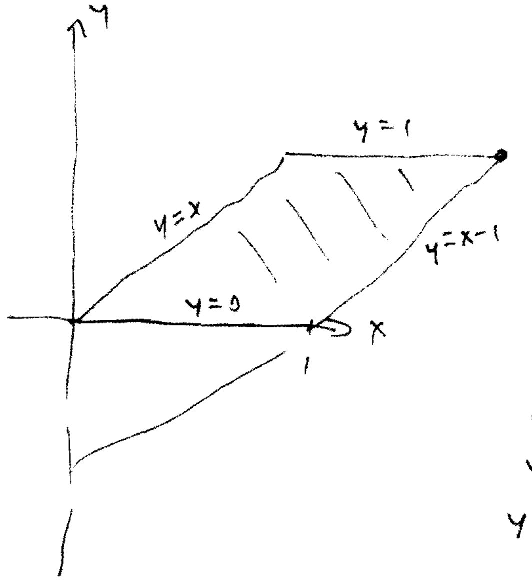
6. (9 points total) Let R be the parallelogram bounded by the lines $y = x$, $y = x - 1$, $y = 0$ and $y = 1$. Consider the integral

$$I = \int_R y(x-y)^{324}\, dA.$$

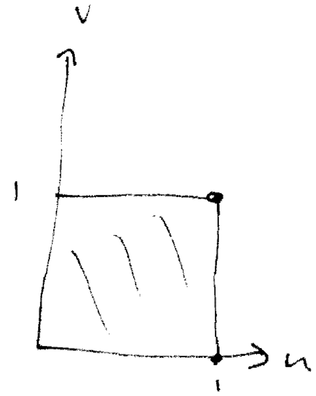
Use the change of variables $x = u + v$, $y = v$. Set up (but do not solve) an integral for I in terms of u and v.

$$f(x,y) = y\,(x-y)^{324}$$
$$= v \cdot u^{324}$$

$$x - y = (u+v) - (v) = u$$



$\longleftrightarrow$



$y = 0 \longrightarrow v = 0$

$y = 1 \longrightarrow v = 1$

$y = x \longrightarrow v = u + v$

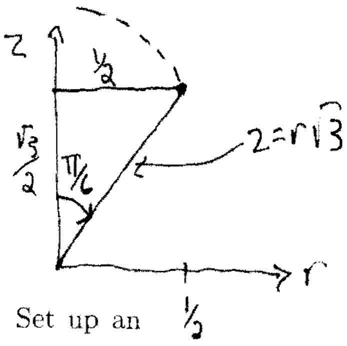
$\qquad u = 0$

$y = x - 1 \longrightarrow v = u + v - 1$

$\qquad u = 1$

$$\boxed{\int_0^1 \int_0^1 v u^{324} \cdot (1) \cdot du\, dv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0$$
$$= 1$$

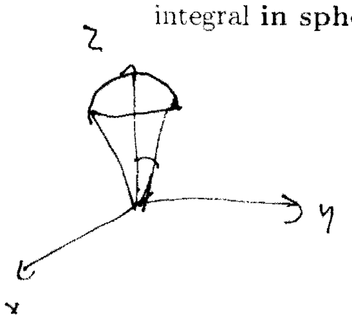


7. (5+5=10 points total) Set up but do not evaluate the following integrals.

(a) Let E be the region within the unit sphere, such that $z \geq \sqrt{3(x^2+y^2)}$. Set up an integral **in spherical coordinates** over the region E, of the function

$z \geq r\sqrt{3}$

$$f(x, y, z) = \sqrt{3(x^2+y^2+z^2)}. \quad = \rho\sqrt{3}$$



$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/6$

(see picture)

$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^{1} (\rho\sqrt{3})\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

(b) Set up an integral of the same function f, over the same region E, but **in cylindrical coordinates**.

$f = \sqrt{3(r^2+z^2)}$

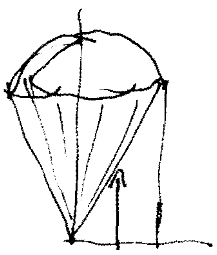
top of sphere: $r^2+z^2=1$

$z = \sqrt{1-r^2}$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \frac{1}{2}$ (from picture)

$r\sqrt{3} \leq z \leq \sqrt{1-r^2}$



$$\int_0^{2\pi}\int_0^{1/2}\int_{z=r\sqrt{3}}^{\sqrt{1-r^2}} \sqrt{3(r^2+z^2)} \, r \, dz \, dr \, d\theta$$