Math 324, Autumn 2011

Midterm 1B Wednesday, October 19th

Name: Solutions

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:________________________

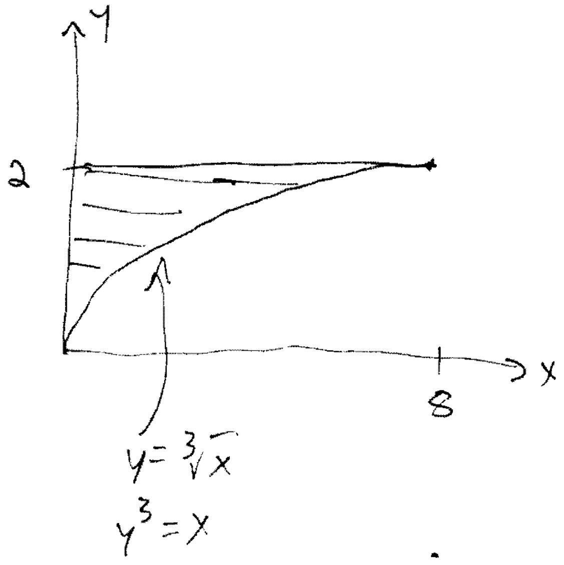
1	5	
2	5	
3	10	
4	3	
5	8	
6	10	
7	9	
EC	4	
Total	50	

- Please check that your exam contains 7 problems on 8 pages.
- Please turn your cell phone OFF and put it away for the duration of the exam.
- Unless otherwise indicated, you must show your work. The correct answer with no supporting work may result in no credit.
- Put your name on your sheet of notes and turn it in with the exam.

GOOD LUCK!

1. (5 points total) Calculate the iterated integral by reversing the order of integration.

$$I = \int_0^8 \int_{\sqrt[3]{x}}^2 e^{y^4}\, dy\, dx.$$



$$I = \int_{y=0}^{y=2} \int_{x=0}^{x=y^3} e^{y^4}\, dx\, dy$$

inside:

$$\int_0^{y^3} e^{y^4}\, d\cancel{x} = y^3 e^{y^4}$$

$$I = \int_0^2 y^3 e^{y^4}\, dy \qquad u = y^4 \qquad \frac{du}{4} = \frac{4y^3}{4} dy$$

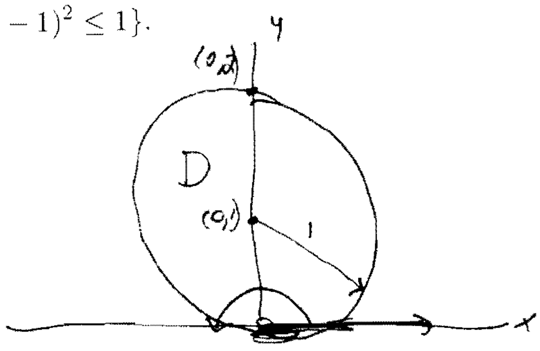
$$= \frac{1}{4}\int e^u\, du = \frac{1}{4} e^{y^4}\Big]_{y=0}^{2} = \frac{1}{4}\left(e^{16} - e^0\right)$$

$$= \boxed{\frac{1}{4}\left(e^{16} - 1\right)}$$

2. (5 points total) Evaluate the following integral.

$r\cos\theta$ in polar

$$I = \int_D x \, dA. \text{ where } D = \{(x, y) \mid x^2 + (y-1)^2 \leq 1\}.$$



D is inside $x^2 + y^2 - 2y + 1 = 1$

$r^2 = 2y$

$r^2 = 2r\sin\theta$

don't worry about $r=0$

$r = 2\sin\theta$

$0 \leq r \leq 2\sin\theta$

$0 \leq \theta \leq \pi$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{r=2\sin\theta} (r\cos\theta)\, r \, dr\, d\theta = \int_0^{\pi} \frac{(2\sin\theta)^3}{3} \cos\theta \, d\theta$$

$$= \int_0^{\pi} \frac{8}{3} \sin^3\theta \cos\theta \, d\theta$$

use $u = \sin\theta$

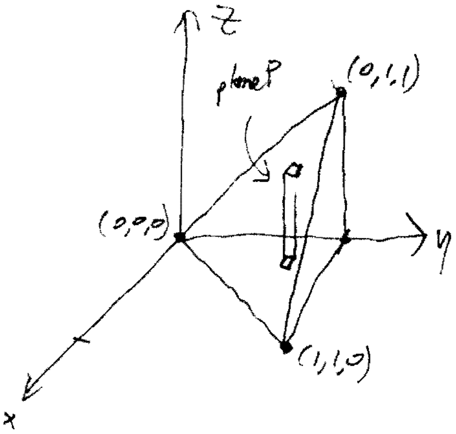
$du = \cos\theta \, d\theta$

$$= \int \frac{8}{3} u^3 du = \frac{8}{3} \cdot \frac{\sin^4\theta}{4} \Bigg]_{\theta=0}^{\pi} = \frac{8}{3}\left[\cancel{\frac{\sin^4(\pi)}{4}}^{0} - \cancel{\frac{\sin^4(0)}{4}}^{0}\right]$$

$$\boxed{0}$$

or easy way:

by symmetry, $I = 0$.



3. (6+4=10 points total)

(a) Let E be the solid tetrahedron with vertices

$$(0,0,0), (0,1,0), (1,1,0), \text{ and } (0,1,1).$$

Set up **but do not evaluate** the integral of $f(x,y,z) = xy^2z$ over E.

equation of plane P: $ax+by+cz=d$

through $(0,0,0)$, $(1,1,0)$, $(0,1,1)$

$d=0$ $\quad a+b=0$ $\quad b+c=0$

pick $a=1$

then $b=-1$

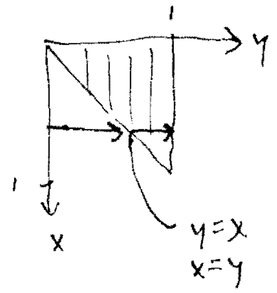
$c=1$

so P is

$x-y+z=0$

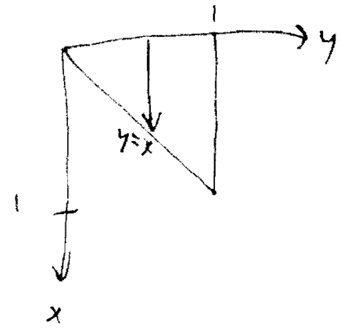
$z=y-x$

$$\boxed{\int_{x=0}^{x=1}\int_{y=x}^{y=1}\int_{0}^{y-x} xy^2z\,dz\,dy\,dx}$$



(b) Set up a different integral of the same function f, over the same tetrahedron E. The order of integration (e.g. $dx\,dy\,dz$) must be different than in the previous problem.

$$\boxed{\int_{y=0}^{y=1}\int_{x=0}^{x=y}\int_{z=0}^{z=y-x} xy^2z\,dz\,dx\,dy}$$



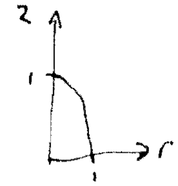
of course, 4 other ways

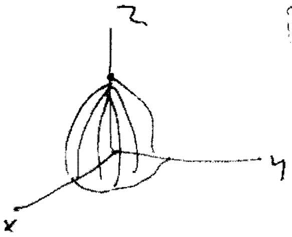
4. (1+1+1=3 points total) Describe in words the surface whose equation is given. Attempt a sketch if you want.

(a) $z = 1 - r^2$.

cylindrical.

no θ.



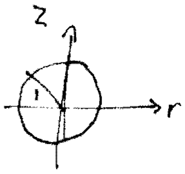


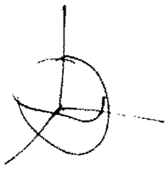
$\boxed{\text{upside down paraboloid}}$

(b) $r^2 + z^2 = 1$.

cylindrical.

no θ.





sphere centered @ origin w/ radius 1

$x^2+y^2+z^2=1$

(c) $\rho = 2\cos\theta \,\sin\phi$.

spherical.

$\rho^2 = 2\rho\cos\theta\sin\phi$

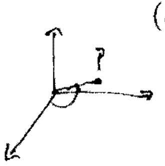
$x^2+y^2+z^2 = 2x \longrightarrow x^2-2x+1+y^2+z^2=1$

$(x-1)^2+y^2+z^2=1$

sphere w/ radius 1 centered @ $(1,0,0)$

5. (2+2+2+2=8 points total)

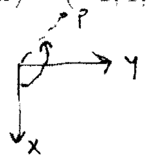
(a) Convert the point $(x, y, z) = (-1, 1, -1)$ from rectangular to cylindrical coordinates.



$z=-1$

$\theta = \frac{3\pi}{4}$

$r = \sqrt{2}$

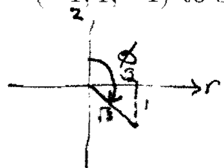


$\left(\sqrt{2}, \frac{3\pi}{4}, -1\right)$

(b) Convert the same point $(x, y, z) = (-1, 1, -1)$ to spherical coordinates.

$\theta = \frac{3\pi}{4}$ from (a)

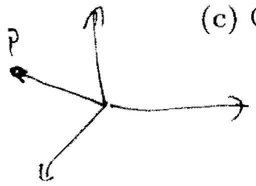
$\rho = \sqrt{1+1+1} = \sqrt{3}$



$\phi = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\phi = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(c) Convert the point $(\rho, \theta, \phi) = (2, \frac{3\pi}{2}, \frac{\pi}{3})$ from spherical to rectangular coordinates.



$x=0 = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{2}\right)$

$y = 2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{2}\right) = -\sqrt{3}$

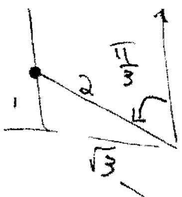
$z = 2\cos\left(\frac{\pi}{3}\right) = 1$

$\left(0, -\sqrt{3}, 1\right)$

(d) Convert the same point $(\rho, \theta, \phi) = (2, \frac{3\pi}{2}, \frac{\pi}{3})$ to cylindrical coordinates.

$z=1$ from (c)

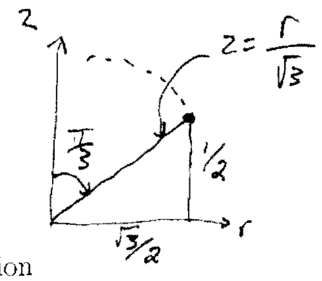
$\theta = \frac{3\pi}{2}$



$r = \sqrt{3}$

$\left(\sqrt{3}, \frac{3\pi}{2}, 1\right)$

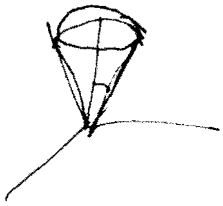
6. (5+5=10 points total) Set up but do not evaluate the following integrals.



(a) Let E be the region within the unit sphere, such that

$$z \geq \sqrt{\frac{x^2+y^2}{3}}. = \frac{r}{\sqrt{3}}$$

Set up an integral **in spherical coordinates** over the region E, of the function



$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/3$

$$f(x, y, z) = \sqrt{\frac{x^2+y^2+z^2}{3}}. = \frac{\rho}{\sqrt{3}}$$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \left(\frac{\rho}{\sqrt{3}}\right)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) Set up an integral of the same function f, over the same region E, but **in cylindrical coordinates**.

$f = \frac{1}{\sqrt{3}}\sqrt{r^2+z^2}$

sphere: $r^2+z^2=1$

$z = \sqrt{1-r^2}$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \frac{\sqrt{3}}{2}$

$\frac{r}{\sqrt{3}} \leq z \leq \sqrt{1-r^2}$

$$\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{r/\sqrt{3}}^{\sqrt{1-r^2}} \sqrt{\frac{r^2+z^2}{3}}\, r \, dz \, dr \, d\theta$$

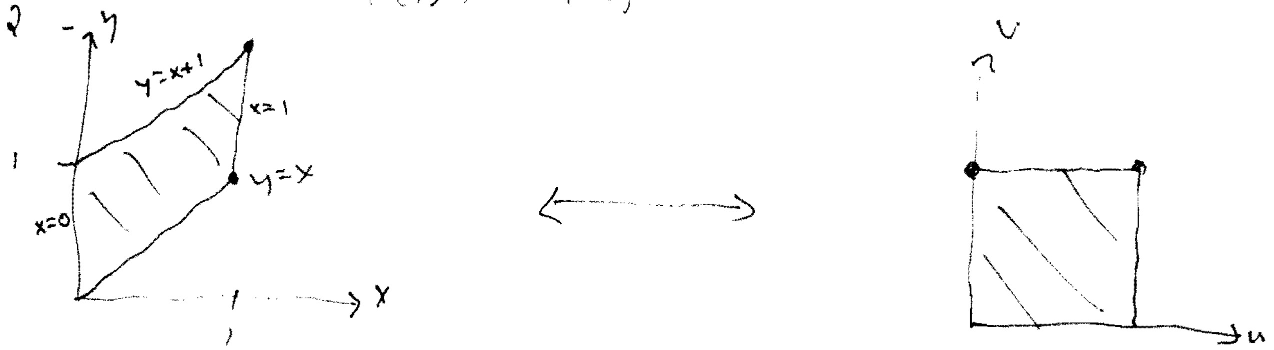
7. (9 points total) Let R be the parallelogram bounded by the lines $y = x$, $y = x + 1$, $x = 0$ and $x = 1$. Consider the integral

$$I = \int_R x(y-x)^{324}\, dA.$$

Use the change of variables $x = u$, $y = u + v$. Set up (but do not solve) an integral for I in terms of u and v.

$$y - x = (u+v) - u = v$$

$$f(x,y) = x(y-x)^{324} = u\,v^{324}$$



$$x = 0 \longrightarrow u = 0$$

$$x = 1 \longrightarrow u = 1$$

$$y = x \longrightarrow u + v = u, \quad v = 0$$

$$y = x+1 \longrightarrow u + v = u + 1, \quad v = 1$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$\boxed{I = \int_0^1 \int_0^1 u\, v^{324} \cdot (1)\, du\, dv}$$