Generalized Ray Transform

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1 Generalized Ray Transform: Setting in two dimensions.

1.1 Family of curves.

We consider the following family of curves

$$\mathbb{R} \ni t \mapsto x = \gamma(t,s,\theta) \in \mathbb{R}^2, \qquad (s,\theta) \in \mathbb{R}\times\mathbb{S}^1. \tag{1}$$

We assume that curves are traveled along with unit speed so that $|\dot\gamma| = 1$. We will make several assumptions on the curves as we proceed.

Define $y = (t,s)$ so that $\gamma = \gamma(y,\theta)$. We **assume** that the map $y \mapsto \gamma(y,\theta)$ is globally invertible for all values of θ with inverse $\tilde\gamma$ so that

$$\gamma(\tilde\gamma(x,\theta),\theta) = x, \quad \tilde\gamma(\gamma(y,\theta),\theta) = y.$$

We denote the inverse function

$$\tilde\gamma(x,\theta) = (t(x,\theta), s(x,\theta)). \tag{2}$$

The function $s(x,\theta)$ provides the unique curve to which x belongs for a fixed θ, i.e., $x \in \gamma(\mathbb{R}, s(x,\theta),\theta)$.

For the Radon transform corresponding to integration along straight lines, we have

$$\gamma(t,s,\theta) = s\theta^\perp + t\theta, \qquad s(x,\theta) = x\cdot\theta^\perp.$$

This corresponds to a parameterization of the set of lines where θ is the vector tangent to the line and $\theta^\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\theta$ is the vector θ rotated by $\frac{\pi}{2}$. This notation generalizes to the multi-dimensional cases, where θ still parameterizes the direction of the lines. This should not be confused with the Radon transform, which corresponds to integration along hyperplanes (thus lines in dimension $n = 2$). For the Radon transform, θ is typically the vector orthogonal to the hyperplane.

1.2 Generalized Ray Transform.

The generalized Ray transform (GRT) is then the integral of functions over curves $(s,\theta) \mapsto \gamma(t,s,\theta)$:

$$\begin{aligned} Rf(s,\theta) &= \int_{\mathbb{R}} f(\gamma(t,s,\theta))dt = \int_{\mathbb{R}^2} f(\gamma(t,s_0,\theta))\delta(s-s_0)ds_0dt \\ &= \int_{\mathbb{R}^2} f(x)\delta(s-s(x,\theta))J(x,\theta)dx, \end{aligned} \tag{3}$$

where $J(x,\theta)$ is the (uniformly positive) Jacobian of the transformation $x \to \tilde\gamma(x,\theta)$ at fixed θ:

$$J(x,\theta) := \Big|\frac{d\tilde\gamma}{dx}\Big|(x,\theta) = \Big|\frac{ds_0dt}{dx}\Big|(x,\theta). \tag{4}$$

Exercise 1.1 *Check the change of variables in detail.*

More generally, we considered weighted GRT of the form

$$R_w f(s,\theta) \;=\; \int_{\mathbb{R}} f(\gamma(t,s,\theta))w(t,s,\theta)dt, \tag{5}$$

where $w(y,\theta)$ is a given, positive, weight. Such integrals are of the same form as before:

$$\begin{aligned} R_w f(s,\theta) &= \int_{\mathbb{R}} f(\gamma(t,s,\theta))w(t,s,\theta)dt = \int_{\mathbb{R}^2} f(\gamma(t,s_0,\theta))\delta(s-s_0)w(t,s_0,\theta)ds_0dt \\ &= \int_{\mathbb{R}^2} f(x)\delta(s-s(x,\theta))J_w(x,\theta)dx, \end{aligned}$$

with a different expression for $J(x,\theta)$:

$$J(x,\theta) \equiv J_w(x,\theta) := \Big|\frac{d\tilde\gamma}{dx}\Big|(x,\theta)w(\tilde\gamma(x,\theta),\theta). \tag{6}$$

To simplify, we shall use the notation $J(x,\theta)$ rather than $J_w(x,\theta)$.

The objective of this section is to obtain a parametrix for the weighted GRT. Injectivity of the transform for weights close to the constant weight is obtained in the section on kinematic transforms (usint the Mukhometov technique). Thus generally, we consider an operator of the form

$$R_J f(s,\theta) = \int_{\mathbb{R}^2} f(x)\delta(s-s(x,\theta))J(x,\theta)dx, \tag{7}$$

where $J(x,\theta)$ is a smooth, uniformly positive, and bounded weight.

1.3 Adjoint operator and rescaled Normal operator

When $J \equiv 1$ and $s(x,\theta) = x\cdot\theta^\perp$, then R_J is the standard Radon transform in two dimensions. We then have the inversion formula

$$I = \frac{1}{4\pi} R_J^* \Lambda R_J, \qquad \Lambda = H\frac{\partial}{\partial s}.$$

We thus see the need to introduce the Riesz operator Λ and the adjoint operator R_J^*. In curved geometries, however, no explicit formula such as the one given above can be obtained in general. We no longer have access to the Fourier slice theorem, which uses the invariance by translation of the geometry in the standard Radon transform.

The adjoint operator (AGRT) for the L^2 inner product on $\mathbb{R}\times\mathbb{S}^1$ is defined as

$$R_K^* g(x) = \int_{\mathbb{S}^1} g(s(x,\theta),\theta)K(x,\theta)d\theta = \int_{\mathbb{R}\times\mathbb{S}^1} g(s,\theta)K(x,\theta)\delta(s-s(x,\theta))d\theta ds. \tag{8}$$

The "normal" operator is thus given by

$$R_K^* R_J f(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1} f(y)K(x,\theta)J(y,\theta)\delta(s(x,\theta)-s(y,\theta))dyd\theta. \tag{9}$$

Exercise 1.2 *Check this.*

We need to introduce $H\partial_s$ to make the operator invertible from L^2 to L^2 as in the case of the standard Radon transform. A simple way to do so is to recast the operators as Fourier integral operators (FIOs) as follows. We formally recast the GRT and AGRT as the following oscillatory integrals

$$\begin{aligned} R_J f(s,\theta) &= \int_{\mathbb{R}^2\times\mathbb{R}} f(x)e^{i(s-s(x,\theta))\sigma} J(x,\theta)\frac{dxd\sigma}{2\pi} \\ R_K^* g(x) &= \int_{\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}} g(s,\theta)e^{-i(s-s(x,\theta))\sigma} K(x,\theta)\frac{dsd\theta d\sigma}{2\pi}. \end{aligned} \tag{10}$$

We then introduce the Riesz operator $\Lambda = H\partial_s$ given by the Fourier multiplier $|\sigma|$ in the Fourier domain:

$$\Lambda f(s) = H\partial_s f(s) = \mathcal{F}^{-1}_{\sigma\to s}|\sigma|\mathcal{F}_{s\to\sigma} f(s).$$

We thus have

$$\Lambda R_J f(s,\theta) = \int_{\mathbb{R}^2\times\mathbb{R}} f(x)|\sigma|e^{i(s-s(x,\theta))\sigma} J(x,\theta)\frac{dxd\sigma}{2\pi}.$$

Exercise 1.3 *Check this.*

The "normal" operator for the weights J and K is then defined as

$$Ff(x) := R_K^*\Lambda R_J f(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1\times\mathbb{R}} f(y)|\sigma|e^{i(s(y,\theta)-s(x,\theta))\sigma} K(x,\theta)J(y,\theta)\frac{dyd\sigma d\theta}{2\pi}. \tag{11}$$

It remains to analyze such an operator. We shall have two main objectives: (i) find the appropriate value for $K(x,\theta)$ that makes F an approximation of identity; and (ii) see how R_J can be inverted in some specific cases.

We first observe that $Ff(x)$ is real valued if we choose J and K to be real-valued. The contribution from $\sigma > 0$ is the same as that from $\sigma < 0$ by complex-conjugating (11). Thus,

$$Ff(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1\times\mathbb{R}_+} f(y)e^{i(s(y,\theta)-s(x,\theta))\sigma} K(x,\theta)J(y,\theta)\frac{dy\,\sigma d\sigma d\theta}{\pi}. \tag{12}$$

The variables (θ,σ) may be recast as $\zeta = \sigma\theta$ in $\mathbb{R}^2$ so that $\hat\zeta = \theta$ and $|\zeta| = \sigma$. We then recast the above operator as

$$Ff(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(y)e^{i(s(y,\hat\zeta)-s(x,\hat\zeta))|\zeta|} K(x,\hat\zeta)J(y,\hat\zeta)\frac{dyd\zeta}{\pi}. \tag{13}$$

The phase and the amplitude are then defined as

$$\boxed{\phi(x,y,\zeta) = \big(s(x,\hat\zeta) - s(y,\hat\zeta)\big)|\zeta|,} \qquad \boxed{a(x,y,\hat\zeta) = \frac{1}{\pi}K(x,\hat\zeta)J(y,\hat\zeta).} \tag{14}$$

The phase is homogeneous of degree 1 in ζ since $\phi(x,y,t\zeta) = t\phi(x,y,\zeta)$ for $t > 0$. The amplitude is homogeneous of degree 0. We may thus recast the above operator as

$$\boxed{Ff(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i\phi(x,y,\zeta)}a(x,y,\hat\zeta)f(y)dyd\zeta.} \tag{15}$$

Our first objective is to prove that for an appropriate choice of $K(x,\theta)$, then F may be decomposed as $F = I - T$ where T is a compact, smoothing, operator. This provides an approximate inversion to the generalized Radon transform and an iterative exact reconstruction procedure in some cases. In general, however, it seems difficult to show that T does not admit 1 as an eigenvalue. This property is expected to hold generically. But it is not clear (and certainly not known) how to prove it for specific transforms of interest.

A second objective is therefore to look at the operator $N = R_J^*\Lambda R^J$, which is a self-adjoint (normal) operator. We shall show that N is injective in some cases of interest (using the Mukhometov technique) and that $QN = I - T$ for some operator Q and compact operator T. This will allow us to prove that N is an invertible operator in L^2. The generalized transform can then be inverted by means of, e.g., a conjugate gradient algorithm. This provides an explicit reconstruction procedure that is guaranteed to provide the correct inverse.

2 Oscillatory integrals and Fourier Integral Operators

In this section, we collect generic properties on oscillatory integrals and operators of the form (15). References are Fourier Integral Operators I by Lars Hörmander and Fourier integrals in classical analysis by Christopher Sogge.

2.1 Symbols, phases and oscillatory integrals.

The first main player is the classes of **symbols**

Definition 2.1 *We denote by $S^m(X\times\mathbb{R}^N)$ the set of $a \in C^\infty(X\times\mathbb{R}^N)$ s.t. for compact $K \subset X$ and all multi-orders α and β, we have*

$$|D_x^\beta D_\zeta^\alpha a(x,\zeta)| \leq C_{\alpha,\beta,K}(1+|\zeta|)^{m-|\alpha|}, \quad (x,\zeta)\in K\times\mathbb{R}^N. \tag{16}$$

Throughout the notes, X itself will be bounded and a can be chosen of class $C^\infty(\bar{X}\times\mathbb{R}^N)$ with then $K = \bar{X}$ above.

The second main player is the **phase** $\phi(x,\zeta)$ that appears in the following **oscillatory integrals**:

$$I_\phi(au) = \int_{X\times\mathbb{R}^N} e^{i\phi(x,\zeta)} a(x,\zeta)u(x)dxd\zeta, \quad u \in C_0^\infty(X). \tag{17}$$

The phase is assumed to be **positively homogeneous of degree 1 with respect to** ζ, i.e., $\phi(x,t\zeta) = t\phi(x,\zeta)$ for $t > 0$, and that $\phi \in C^\infty$ for $\zeta \neq 0$. Typically, the phase is defined for $|\zeta| = 1$. It is then extended to all values of ζ by homogeneity, except at $\zeta = 0$ where it is not smooth.

These oscillatory integrals need to be defined carefully since the integrand is not Lebesgue-integrable. When ϕ vanishes on open sets, the above integral may simply not be defined at all. However, a reasonable definition can be given when ϕ has **no critical point** in *joint variables* (x,ζ) for $\zeta \neq 0$, i.e., $d\phi \neq 0$ for $\zeta \neq 0$. Here,

$$d\phi = \phi'_x dx + \phi'_\zeta d\zeta.$$

No critical point means $|\phi'_x|^2 + |\phi'_\zeta|^2 > 0$ for $\zeta \neq 0$. By definition, a **phase** is a smooth function that (i) is positively homogeneous of degree one and (ii) has no critical points in the variables (x,ζ) when $\zeta \neq 0$.

By hypothesis, the sum

$$\psi := |\zeta|^2|\phi'_\zeta|^2 + |\phi'_x|^2 > 0$$

for $\zeta \neq 0$ is homogeneous of degree 2 since ϕ'_ζ is homogeneous of degree 0 and ϕ'_x is homogeneous of degree 1 (all in the ζ variable).

Exercise 2.2 *Check this.*

Let $\chi(\zeta) \in C_0^\infty(\mathbb{R}^N)$ so that $\chi = 1$ near $\zeta = 0$ and define the differential operator

$$M = -\alpha \cdot \nabla_\zeta - \beta \cdot \nabla_x + \chi, \qquad \alpha = \frac{i(1-\chi)}{\psi}|\zeta|^2 \nabla_\zeta \phi, \quad \beta = \frac{i(1-\chi)}{\psi} \nabla_x \phi.$$

We verify that M is a differential operator with smooth coefficients and that

$$Me^{i\phi} = e^{i\phi}$$

Exercise 2.3 *Verify this in detail.*

Let us then define the transpose differential operator

$$L = M^t = \alpha \cdot \nabla_\zeta + \beta \cdot \nabla_x + \chi + \nabla_\zeta \cdot \alpha + \nabla_x \cdot \beta.$$

Assuming that $a(x,\zeta)$ vanishes for large ζ, we can integrate (17) by parts and obtain that

$$I_\phi(au) = \int_{X \times \mathbb{R}^N} e^{i\phi(x,\zeta)} L^k\big(a(x,\zeta)u(x)\big) dx d\zeta, \quad u \in C_0^\infty(X), \quad k = 0, 1, 2, \ldots. \tag{18}$$

The advantage of these integrations by parts is that L^k maps S^m into S^{m-k}.

Exercise 2.4 *Check this latter result as well as the integrations by parts in* (18).

Once $m - k < -N$, then the integral is well defined as a classical Lebesgue integral.

Exercise 2.5 *Check this.*

We can then pass by continuity to the definition of the integral when $a(x,\zeta)$ does not vanish for large ζ. This is our *definition* of the oscillatory integral (which we can check does not depend on the choice of χ or k).

Note that $A : u \mapsto I_\phi(au)$ is then defined as a distribution. This is in fact a distribution of order $\leq k$ if $a \in S^m$ and $m - k < -N$; see also (20) below.

We also verify using the same integrations by parts that

$$I_\phi(au) = \lim_{\varepsilon\to 0} \int e^{i\phi(x,\zeta)} a(x,\zeta)\chi(\varepsilon\zeta)u(x)dxd\zeta, \quad u \in C_0^\infty(X), \tag{19}$$

for $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $\chi(0) = 1$. If a and ϕ depend on a **parameter** t in a continuous manner, then, using the above characterization as a limit when $\varepsilon \to 0$, the integral is also continuous in that parameter:

$$t \to I_\phi(au, t) = \int_{X\times\mathbb{R}^N} e^{i\phi(x,\zeta,t)} a(x,\zeta,t)u(x)dxd\zeta$$

is continuous. We can thus differentiate with respect to these parameters under the integral sign. We also have Fubini theorems to exchange the order of integrations.

2.2 Parameterized oscillatory integrals

This construction allows us to introduce the following **parameterized oscillatory integrals**. We denote by X_ϕ the open subset of X such that $\zeta \mapsto \phi(x,\zeta)$ has no critical point $\zeta \neq 0$. This means that

$$|\phi'_\zeta|^2(x,\zeta) > 0, \quad x \in X_\phi,\ \zeta \neq 0.$$

This is obviously more constraining than the previous constraint in the variable (x,ζ) since we have less partial derivatives to be non-vanishing. Seeing x as a parameter, we can then define the oscillatory integral as

$$I_\phi(a,u) = \int_X A(x)u(x)dx,\ u \in C_0^\infty(X_\phi), \quad A(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\zeta)} a(x,\zeta)d\zeta,\ x \in X_\phi. \tag{20}$$

Then $A(x)$ is continuous and even a function in $C^\infty(X_\phi)$.

Let us see some consequences on the singularities of the distribution A. The same integrations by parts as above show that for A considered as a distribution, we have

$$\text{sing supp} A \subset \{x \in X;\ \phi'_\zeta(x,\zeta) = 0 \text{ for some } \zeta \neq 0\}.$$

Exercise 2.6 *Check this.*

More precisely, we have that for a symbol vanishing in some conic neighborhood of the set

$$C = \{(x,\zeta), x \in X, \zeta \in \mathbb{R}^N\backslash\{0\},\ \phi'_\zeta(x,\zeta) = 0\},$$

then the distribution $A(x)$ defined by $u \mapsto I_\phi(au)$ is a C^∞ function.

By conic neighborhood of ζ, we mean a set of vectors such that ξ belongs to the neighborhood when $|\hat{\xi} - \hat{\zeta}| < \delta$ for some $\delta > 0$. Here and below, we denote by $\hat{\xi} = \frac{\xi}{|\xi|}$ the direction of ξ.

Exercise 2.7 *Check this by integrations by parts.*

This result is useful but that cannot be applied directly to operators. For this, we need to split the variables x into the set of variables (x, y), where x denotes the parameters in which the operator is defined and y denotes the variables of integration of the function on which the operator acts.

2.3 Definition of Fourier Integral Operators.

Consider $X \times Y \subset \mathbb{R}^{n_X} \times \mathbb{R}^{n_Y}$ and $\zeta \in \mathbb{R}^N$. Then $\phi(x, y, \zeta)$ is positively homogeneous of degree 1 in ζ. For a symbol $a \in S^m$, we consider the operator

$$Au(x) = \int_{Y \times \mathbb{R}^N} e^{i\phi(x,y,\zeta)} a(x, y, \zeta) u(y) dy d\zeta, \quad u \in C_0^\infty(Y),\ x \in X. \tag{21}$$

If ϕ does not have any critical point as a function of all variables (x, y, ζ), then we have seen that the integral was well defined as a distribution (after multiplication by $v(x) \in C_0^\infty(X)$ and integration on X).

If for each fixed x, $\phi(x, y, \zeta)$ does not have critical points as a function of (y, ζ), then (21) is well defined. Moreover, A is a continuous map from $C_0^k(Y)$ to $C^j(X)$ if

$$m + N + j < k,$$

using the terminology of the previous sections.

Exercise 2.8 *Check this.*

The same occurs for the adjoint operator A^* by exchanging the roles of x and y.

Let R_ϕ the open set of points (x, y) such that $\phi'_\zeta \neq 0$ for $\zeta \neq 0$. Then

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\zeta)} a(x, y, \zeta) d\zeta, \qquad (x, y) \in R_\phi,$$

is a Schwartz kernel for A and defines a function in $C^\infty(R_\phi)$. If $R_\phi = X \times Y$, then K_A defines a map A continuous from $\mathcal{E}'(Y)$ to $C^\infty(X)$.

Example 1. Let us consider the phase function $\phi(x, y, \zeta) = (x - y) \cdot \zeta$. We obtain that R_ϕ is the complement of the diagonal $x = y$.

Example 2. For the GRT with $\phi(s, \theta, x, \sigma) = \sigma(s - s(x, \theta))$, then R_ϕ is the complement of the set $s = s(x, \theta)$. In other words, a singularity at a point x can a priori generate singularities along all the points of the curve in the (s, θ) plane given by $s = s(x, \theta)$. (Note that, here "(s, θ)" plays the role of "x", "x" plays the role of "y", and "σ" plays the role of "ζ".) For the RT a singularity at $(0, x_2)$ for instance generates the curve $s = x_2 \cos\theta$. Similarly, for the adjoint operator, a singularity at (s, θ) becomes a potential singularity along the line $s = x \cdot \theta^\perp$ (a curve for the GRT). It looks as if a singularity at one point x propagates to singularities everywhere after application of R^*R. That this is not the case shows that the singular support is not sufficient to fully characterize the propagation of singularities: the propagation of singularities for the Radon transform requires the notion of singularities in the phase space, in other words, requires the introduction of the Wave Front Set.

Nonetheless, we have the following result. We call ϕ an operator phase function if for each x (or y) it has no critical point in (y, ζ) (or (x, ζ)). Let C_ϕ be the complement of R_ϕ, i.e., the projection on $X \times Y$ of

$$C = \{(x, y, \zeta) \in X \times Y \times \mathbb{R}^N \backslash \{0\},\ \phi'_\zeta(x, y, \zeta) = 0\}.$$

In other words, $(x, y) \in C_\phi$ if (and only if) there exists $\zeta \neq 0$ such that $(x, y, \zeta) \in C$. Then we have:

$$\text{sing supp} Au \subset C_\phi \text{supp} u, \quad u \in \mathcal{E}'(Y).$$

Here, we have defined

$$C_\phi K = \{x,\ (x,y) \in C_\phi \text{ for some } y \in K\}.$$

In fact we can write $u = v + w$ with v supported in the vicinity of the singular support of u and w smooth. This allows us to obtain the more refined result:

$$\boxed{\text{sing supp} Au \subset C_\phi \text{sing supp}\, u, \quad u \in \mathcal{E}'(Y).} \tag{22}$$

The latter result is sometimes satisfactory, sometimes not. For the phase function $\phi = (x-y)\cdot\zeta$, we obtain that the singularities of Au are included in the set of singularities of u. In some sense, this is satisfactory as it states that singularities cannot propagate in the x variable.

For the phase $\phi(s,\theta,x,\sigma) = \sigma(s-s(x,\theta))$, however, the results are not satisfactory as they imply that singularities at a point x can propagate to singularities at any point (s,θ) and it therefore looks like singularities are spread very wildly by the Radon transform. The notion of wave front sets will refine this statement and show that there is in fact a one-to-one (in fact one-to-two) correspondence between properly defined singularities before and after application of the Radon transform.

3 Pseudo-differential operators and GRT

We now come back to the definition of the operator $Ff(x)$ appearing in the analysis of the GRT. Although the phase appears to be complicated, it is in fact essentially of the form $\phi(x,y,\xi) = (x-y)\cdot\xi$ after an appropriate change of variables.

3.1 Absence of singularities away from the diagonal $x = y$

We start with a first step showing that $Ff(x)$ singularities emitted at a point y do not propagate away from that point under reasonable assumptions on the phase.

Let us **assume** that

$$\phi'_\zeta(x,y,\zeta) = 0 \quad \textit{implies that} \quad x = y.$$

Exercise 3.1 *Show that the above constraint is satisfied for the Radon transform.*

Let then $\chi_0(x,y) = 1$ when $x = y$ and supported in the vicinity of $x = y$ in the sense that $\chi_0(x,y) = 0$ when $|x-y| > \delta$ for δ to be chosen arbitrarily small. Also we assume that $\chi_0 \in C^\infty(X \times Y)$. Let then

$$Ff(x) = F_0 f(x) + F_1 f(x), \quad F_0 f(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(y) e^{i\phi(x,y,\zeta)} a(x,y,\hat{\zeta})\chi_0(x,y) dy d\zeta. \tag{23}$$

Let us show that $F_1 := F - F_0$ is a regularizing operator. This is a repeat of what we saw in section 2.3 but we shall do the derivation in detail nonetheless. We recall that $\chi(\zeta)$ is a smooth function with compact support in $\mathbb{R}^2$. Since $\phi'_\zeta(x,y,\zeta) \neq 0$ is homogeneous of degree 0, it is bounded from below by a positive constant uniformly in ζ and uniformly in x and y for compact domains X and Y on the support of $1-\chi_0(x,y)$. We then define

$$b_j(x,y,\zeta) = \frac{-i(1-\chi)\partial_{\zeta_j}\phi}{|\phi'_\zeta|^2},$$

which is homogeneous of degree 0 and bounded. We verify that

$$L^t e^{i\phi} := \big(b_j\partial_{\zeta_j} + \chi\big)e^{i\phi} = e^{i\phi}$$

where L^t is a first-order differential operator adjoint to the first-order differential operator L (as we have done before) so that

$$F_1 f(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(y)e^{i\phi(x,y,\zeta)}L^k(a(1-\chi_0))(x,y,\zeta)dyd\zeta, \qquad k\in\mathbb{N}.$$

Since a is bounded and smooth in ζ (in fact homogeneous of degree 0), χ is compactly supported, and b_j are homogeneous of degree 0, we verify that $L^k a$ is bounded by $|\zeta|^{-k}$. Thus for $k\geq 3$, the above integral is absolutely continuous in dimension $n=2$ since $|\zeta|^{-3}|\zeta|d|\zeta|$ is integrable on $(1,\infty)$. Now, we can differentiate the above expression with respect to x. Each differentiation brings a contribution ϕ'_x homogeneous of degree 1 in ζ. If we differentiate j times and choose $k=j+3$, then the integral is absolutely convergent in ζ again. With $j=m$, we observe that F_1 maps $L^2(Y)$ to $H^m(X)$ for all $m\in\mathbb{N}$ and hence is certainly compact by Sobolev imbedding. These are very similar calculations to those of Exercise 2.8.

This calculation shows that the singular support of $Ff(x)$ is included in the support of f, and more precisely in the singular support of f by writing $f=f_1+f_2$ with f_2 smooth and f_1 supported in an arbitrary small neighborhood of the singular support of f. Indeed, by sending $j\to\infty$ and $\delta\to 0$, we observe that $Ff(x)$ is of class C^∞ away from the support of f. Thus, as in (22),

$$\operatorname{sing\,supp} Ff \subset \operatorname{sing\,supp} f. \tag{24}$$

Here, this is a satisfactory result, stating that the singularities of Ff are at the right place. It remains to show that all the singularities of f are captured by F and that F is invertible in some sense.

3.2 Change of variables and phase $(x-y)\cdot\xi$

Let us now return to the analysis of F_0. We want to show that the latter operator is a pseudo-differential operator, namely that after an appropriate change of variables, the phase $\phi(x,y,\zeta)$ can in fact be recast as $(x-y)\cdot\xi$. Let us consider the identity

$$s(x,\hat\zeta) - s(y,\hat\zeta) = (x-y)\cdot\nabla_x s(z(x,y,\hat\zeta),\hat\zeta) = s'_{x_j}(z(x,y,\hat\zeta),\hat\zeta)(x_j-y_j),$$

for some smooth function $z(x,y,\hat\zeta)$. Then we define the change of variables $\xi=\xi(\zeta)$ at fixed y and x:

$$\xi = s'_x(z(x,y,\hat\zeta),\hat\zeta)|\zeta| := h(\hat\zeta;x,y)|\zeta| := \varphi(\zeta). \tag{25}$$

We need this change of variables to be well defined in the vicinity of $x = y$ since the operator F has been replaced by F_0 whose kernel is concentrated in the vicinity of $x = y$. We verify that

$$L(x, y, \zeta) := \Big|\frac{d\xi}{d\zeta}\Big|(x, y, \zeta) = \det(h, h')(x, y, \hat{\zeta}). \tag{26}$$

Here, h' is the derivative of the vector h in the variable ϕ where $\hat{\zeta} = (\cos\phi, \sin\phi)$.

Exercise 3.2 *Check these calculations in detail and show that the determinant is indeed homogeneous of degree* 0 *in* ζ.

We **assume** that $|L(x, y, \zeta)| \geq c_0 > 0$ is uniformly positive for $(x, y) \in X \times Y$ compact domains and $|x - y| < \delta$. For δ sufficiently small, it is thus sufficient to **assume** that

$$\Big|\frac{d\xi}{d\zeta}\Big|(x, x, \hat{\zeta}) = \det(h, h')(x, x, \hat{\zeta}) \geq 2c_0 > 0. \tag{27}$$

This is a local property of invertibility. However, we need to **assume** that $\xi = \varphi(\zeta)$ is a *global* change of variables in the "fiber" variable. In other words, we **assume** that φ is a global diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ with inverse φ^{-1} for fixed values of x and y. Note that since φ is homogeneous of degree 1 so that $\lambda\xi = \varphi(\lambda\zeta)$, then φ^{-1} is also homogeneous of degree 1. As a consequence, $\zeta = |\xi|\varphi^{-1}(\hat{\xi})$ and

$$\hat{\zeta} = \frac{\varphi^{-1}(\hat{\xi})}{|\varphi^{-1}(\hat{\xi})|} := \hat{\zeta}(x, y, \hat{\xi}).$$

Exercise 3.3 *Check that* $\xi = \zeta^\perp$ *and that* $L(x, y, \zeta) = 1$ *when* $\gamma(t, x, \theta) = s\theta^\perp + t\theta$.

With this we recast the operator F_0 as

$$F_0 f(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi} M(x, y, \hat{\xi}) dy d\xi, \tag{28}$$

with

$$M(x, y, \hat{\xi}) = K(x, \hat{\zeta}(x, y, \hat{\xi}))J(y, \hat{\zeta}(x, y, \hat{\xi}))L(x, y, \hat{\zeta}(\hat{\xi}))\chi_0(x, y).$$

Note that $M(x, y, \hat{\xi})$ is a symbol of class $S^0(X \times Y \times \mathbb{R}^N)$ for $N = n = 2$.

Operators of the form

$$Pf(x) = \int_{Y\times\mathbb{R}^n} f(y) e^{i(x-y)\cdot\xi} a(x, y, \xi) dy d\xi, \tag{29}$$

with $a \in S^0(X \times Y \times \mathbb{R}^n)$ are called **pseudo-differential operators** (ΨDOs) of order zero. When $a(\xi)$ is a polynomial, then these are *differential* operators. We have seen that F could be written as the sum of a ΨDO and a smoothing (compact) operator.

3.3 Choice of a parametrix.

In the above construction, J is given by the problem of interest while $K = K(x, \theta)$ is a kernel that we can choose. One way to address the inversion of R_J is to choose $K(x, \theta)$ so that the above operator is as close to the identity operator as possible. We have already shown that all singularities of the kernel of the above oscillatory integral were located on the diagonal $x = y$. We thus define

$$K(x, \hat{\zeta}) = \frac{1}{(2\pi)^2 J(x, \hat{\zeta}) L(x, x, \hat{\zeta})}. \tag{30}$$

With this, we find that $M(x, x, \hat{\xi}) = (2\pi)^{-2}$. In other words, we have

$$\begin{aligned} F_0 f(x) &= f(x) - T_0 f(x), \\ T_0 f(x) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi} \big(M(x, x, \hat{\xi}) - M(x, y, \hat{\xi}) \big) dy d\xi. \end{aligned} \tag{31}$$

We shall prove that T_0 is a compact operator. This will show that $R_K^* \Lambda$ is an inverse of R_J up to a remainder that is a compact operator, in other words that $R_K^* \Lambda R_J = I - T$ where T is a compact operator since we already know that F_1 is a compact operator.

3.4 Proof of smoothing by one derivative

As an operator from a bounded domain Y to a bounded domain X, T_0 is a compact operator in L^2, and in fact an operator mapping $L^2(Y)$ to $H^1(X)$ of these respective domains. This will be proved by showing that T_0 and $\partial_{x_j} T_0$ for $j = 1, 2$ are bounded operators from $L^2(Y)$ to $L^2(X)$.

We consider the operators $\partial_{x_j} T_0$ first and calculate:

$$\partial_{x_j} T_0 f(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi} i\xi_j (x - y) \cdot \nabla_x M(x, \tau(x, y, \hat{\xi}), \hat{\xi}) dy d\xi + T_{00} f(x),$$

for some operator $T_{00} f(x)$ with a symbol in $S^0(X \times Y \times \mathbb{R}^2)$ (which will then be bounded from L^2 to L^2 as we shall see in the next section; check the details as an exercise) and for some smooth function $\tau(x, y, \hat{\xi})$ so that the components of $\nabla_x M(x, \tau(x, y, \hat{\xi}), \hat{\xi})$ belong to $S^0(X \times Y \times \mathbb{R}^N)$.

Let $\chi(\xi)$ still be the compactly supported, smooth function in $\mathbb{R}^2$ such that $\chi(0) = 1$. The above integrand is multiplied by $\chi + (1 - \chi)$. The contribution χ clearly generates a smooth, bounded, contribution. The contribution $(1 - \chi)$ may be recast as a sum of terms of the form

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi} \xi_j (x - y)_k M_k(x, y, \xi) dy d\xi = - \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi} \xi_j \partial_{\xi_k} M_k(x, y, \xi) dy d\xi,$$

for some smooth and bounded functions $M_k(x, y, \xi)$, which are in fact homogeneous of degree 0 for ξ outside the support of $\chi(\xi)$. Now we observe that $\xi_j \partial_{\xi_k} M_k(x, y, \xi) = a_{jk}(x, y, \xi)$ is a symbol in $S^0(X \times Y \times \mathbb{R}^2)$ and is in fact homogeneous of degree 0 for ξ outside the support of $\chi(\xi)$. Indeed, we have in polar coordinates in two dimensions the following explicit expression:

$$\nabla_\xi \psi(\hat{\xi}) = \frac{\partial}{\partial |\xi|} \psi(\hat{\xi}) e_{|\xi|} + \frac{1}{|\xi|} \frac{\partial}{\partial \hat{\xi}} \psi(\hat{\xi}) e_{\hat{\xi}} = \frac{1}{|\xi|} \frac{\partial}{\partial \hat{\xi}} \psi(\hat{\xi}) e_{\hat{\xi}}.$$

So we are faced with showing that an oscillatory operator of the form (29) with a bounded amplitude $a(x,y,\xi)$, homogeneous of degree 0 for large ξ, is bounded from $L^2(Y)$ to $L^2(X)$.

3.5 Boundedness of ΨDOs of order 0 in the L^2 sense

Consider the operator

$$Pf(x) = \int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}a(x,y,\hat{\xi})dyd\xi, \tag{32}$$

with $a(x,y,\hat{\xi}) \in S^0(X\times Y\times\mathbb{R}^n)$. Here, we are in the simplified setting where a is a symbol of order 0 that also turns out to be homogeneous of degree 0. Since the integral over any ball in ξ generates a clearly bounded contribution, what we mean is that $a(x,y,\hat{\xi})$ is homogeneous of degree 0 for $|\xi|$ sufficiently large.

Lemma 3.4 *Let X and Y be compact domains in $\mathbb{R}^n$. Then the operator P in* (32) *is a bounded operator from $L^2(Y)$ to $L^2(X)$ with a constant C independent of $f\in L^2(Y)$ such that*

$$\|Pf\|_{L^2(X)} \leq C\|f\|_{L^2(Y)}. \tag{33}$$

Proof. We first need to separate low frequencies from high frequencies since we have a problem of regularity at $\xi=0$. Let $\chi(\xi)$ be smooth, compactly supported and so that $\chi(0)=1$ and define

$$P = P_0 + P_1, \qquad P_0f(x) = \int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}(1-\chi(\xi))a(x,y,\hat{\xi})dyd\xi,$$

with P_1 clearly bounded from $L^2(Y)$ to $L^2(X)$. Define $a_\chi(x,y,\xi) = (1-\chi(\xi))a(x,y,\hat{\xi})$, noting that for $|\xi|$ sufficiently large, then a_χ is a function of $\hat{\xi}$ since $\chi=1$. For ξ sufficiently small, then a_χ vanishes.

By Taylor expansion, we write

$$a_\chi(x,y,\xi) = \sum_{|\alpha|<k}(y-x)^\alpha a_\alpha(x,x,\xi) + \sum_{|\alpha|=k}(y-x)^\alpha a_\alpha(x,y,\xi).$$

Note that again all functions a_α are functions of $\hat{\xi}$ for $|\xi|$ sufficiently large and vanish for ξ sufficiently small. Now from

$$\partial_\xi^\alpha e^{i(x-y)\cdot\xi} = i^{|\alpha|}(x-y)^\alpha e^{i(x-y)\cdot\xi},$$

we deduce that

$$\int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}(y-x)^\alpha a_\alpha(x,y,\xi)dyd\xi = \int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}i^{|\alpha|}\partial_\xi^\alpha a_\alpha(x,y,\xi)dyd\xi.$$

For $|\alpha|<k$, the above amplitude is a function of (x,x). With these expressions, we find that

$$P_0f(x) = \int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}P_k(x,\xi)d\xi dy + \int_{Y\times\mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi}R_k(x,y,\xi)d\xi dy,$$

where $P_k(x,\xi)$ is a smooth bounded function in both variables, and where $R_k(x,y,\xi)$ is a smooth function bounded by $|\xi|^{-k}$ and vanishing for ξ small.

Exercise 3.5 *Check this. Hint: differentiate k times functions that depend only on $\hat{\xi}$.*

Upon choosing $k = n + 1$, we find that the integral in ξ involving R_k is absolutely convergent. That expression therefore clearly has L^2 norm bounded by that of f by an application of the Cauchy-Schwarz inequality. It remains to address the middle term involving $P_k(x, \xi)$. That this operator is bounded in $\mathcal{L}(L^2)$ is not completely straightforward.

After extending f by 0 on $\mathbb{R}^n \backslash Y$, we recast that term as

$$P_k f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} P_k(x, \xi) d\xi.$$

We write for the two-dimensional case $n = 2$:

$$P_k(x, \xi) = P_k(0, \xi) + \int_0^{x_1} \partial_1 P_k(t_1, 0, \xi) dt_1 + \int_0^{x_2} \partial_2 P_k(0, t_2, \xi) dt_2 + \int_0^{x_1}\int_0^{x_2} \partial^2_{12} P_k(t_1, t_2, \xi) dt_1 dt_2.$$

Exercise 3.6 *Write the corresponding expansion for arbitrary dimension $n \geq 2$.*

Let us define

$$P_{k1} f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \int_0^{x_1} \partial_1 P_k(t_1, 0, \xi) dt_1 d\xi.$$

We observe by Fubini, which holds for oscillatory integrals, that

$$P_{k1} f(x) = \int_0^{x_1} \Big(\int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \partial_1 P_k(t_1, 0, \xi) d\xi \Big) dt_1.$$

But then by Minkowski's integral inequality, we have

$$\|P_{k1} f\|_{L^2(X)} \leq \int_0^{x_1} \Big\| \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \partial_1 P_k(t_1, 0, \xi) d\xi \Big\|_{L^2(X)} dt_1.$$

By using the Plancherel identity and dealing with the other contributions in a similar manner, we obtain that

$$\|P_k f\|_{L^2(X)} \leq C \| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) Q(\xi) d\xi \|_{L^2(X)} \leq C \|\hat{f}(\xi) Q(\xi)\|_{L^2(\mathbb{R}^n)},$$

where $Q(\xi)$ is the maximum over t of the operators $\partial^\alpha P_k(t, \xi)$ for $|\alpha| \leq n$ and $t \in X$. This is a bounded quantity. Moreover, $Q(\xi)$ is bounded since the symbol $P_k(x, \xi)$ is bounded in ξ. This shows the result. □

To come back to our original problem, we have shown that T_0 was bounded from $L^2(Y)$ to $L^2(X)$ and that its partial derivatives in x were also bounded from $L^2(Y)$ to $L^2(X)$. This shows that T_0 is a bounded operator from $L^2(Y)$ to $H^1(X)$. This also shows that F is bounded from $L^2(Y)$ to $L^2(X)$. We summarize these results as follows: We have thus obtained that

$$\|T_0 f(x)\|_{H^1(X)} + \|F f(x)\|_{L^2(X)} \leq C \|f(x)\|_{L^2(Y)}. \quad (34)$$

3.6 Injectivity and implicit inversion formula.

The above result states that $F = I - T$ with T compact from $L^2(X)$ to $L^2(Y)$ since the injection $i : H^1(X) \to L^2(X)$ is compact. The Fredholm alternative thus says that F is *invertible* if it is *injective*. It is not known how to prove injectivity of F in general. All we know is that F is injective if 1 is not an eigenvalue of T. Also, even if 1 is an eigenvalue of T, the operator F can be inverted on the complement of a finite dimensional linear space. But in general, we do not know how to prove that F is invertible. Of course, we know that $T = 0$ for the Radon transform. Therefore, when $\gamma(t, s, \theta)$ is close to $s\theta^\perp + t\theta$, then by continuity, T is of norm less than 1 (we then do not need T to be compact) and then $F = \sum_{k=0}^\infty T^k$.

Something else can be done, however, when we *know* that R_J is injective. In general, we do not know that R_J is injective. It is typically very difficult to prove such injectivity results. In the next section, we shall obtain such an injectivity result by using the Mukhometov technique. Let us define the normal operator

$$Nf(x) = R_J^* \Lambda R_J f(x) \tag{35}$$

as an operator from $L^2(X)$ to $L^2(X)$. Since $\Lambda^{\frac{1}{2}}$ is invertible and self-adjoint, then N is also injective and self-adjoint for the usual inner product on $L^2(X)$. Indeed if A is injective then $A^*Au = 0$ implies that

$$(A^*Au, u) = (Au, Au) = 0$$

so that $u = 0$.

We want to show that N is not only injective but in fact invertible in the L^2 sense. Note that N is no longer of the form $I - T$ with T compact and that we therefore again need to work a bit to get such an invertibility statement.

As a first step using the same microlocal techniques as those of the preceding section, we can show that

$$QN = I - T,$$

for T compact and Q a parametrix of order 0. The construction of Q goes as follows. We can write

$$N = N_0 + T_1,$$

where T_1 is compact and N_0 is given by

$$N_0 f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(y, \xi) f(y) \frac{1}{(2\pi)^n} dy d\xi.$$

Exercise 3.7 *Check that T_1 is indeed a compact operator. Hint: use the same proof showing that T_0 in* (31) *is compact.*

Here $a(y, \xi)$ is a symbol of order 0 uniformly bounded from below by a positive constant by construction of N.

Exercise 3.8 *Give the explicit expression satisfied by $a(y, \xi)$ and show that $a(y, \xi)$ is indeed uniformly bounded from below by a positive constant.*

Define Q as the operator

$$Qf(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \frac{1}{a(x,\xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi. \tag{36}$$

Then we find after some cancellations that

$$QN_0 f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \frac{a(y,\xi)}{a(x,\xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi.$$

Exercise 3.9 *Check the cancellations leading to the preceding calculation. Note that the amplitude depends on y in the definition of N_0 and depends on x in the definition of Q. It is the only "combination" for which the product QN_0 admits a nice compact expression as given above.*

How would you define Q if you wanted an approximate right inverse instead, i.e., an operator Q such that $N_0 Q = I - T$ for T compact ?

This shows that

$$T_2 f(x) = (QN_0 - I) f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \frac{a(y,\xi) - a(x,\xi)}{a(x,\xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi,$$

is a compact operator from $L^2(Y)$ to $H^1(X)$ for the same reasons that T_0 in (31) is compact. This shows the existence of a parametrix Q such that $QN = I - T$ with T compact.

We have seen that such operators Q were bounded from $L^2(X)$ to $L^2(X)$ and therefore deduce that

$$\|f\| \le \|QNf\| + \|Tf\| \le C\|Nf\| + \|Tf\|. \tag{37}$$

Since N is injective, we can in fact show that there exists $C_0 > 0$ such that

$$\|f\| \le C_0 \|Nf\|. \tag{38}$$

Proof. Indeed, assume that there is no such constant so that we can construct f_n such that

$$1 = \|f_n\| = n\|Nf_n\|.$$

Then f_n converges weakly to f in L^2 with then $\|Nf_n\| \to 0$. But then $(Nf_n, f) = (f_n, Nf) \to (f, Nf) = 0$, which implies that $f = 0$ since N is self-adjoint and injective. Since T is compact, we obtain that Tf_n converges to $Tf = 0$. Now (37) implies that

$$1 = \|f_n\| \le C\|Nf_n\| + \|Tf_n\| \le \frac{C}{n}\|f_n\| + o(1) = o(1),$$

which is a contradiction. Here, $o(1)$ means a sequence of real numbers that converges to 0 as $n \to \infty$. This shows the existence of $C_0 > 0$ such that (38) holds. □

This proves that N is invertible with inverse N^{-1} bounded by C_0 (indeed the above estimate proves that N has closed range and trivial kernel so that its range, which is closed, is all of $L^2(X)$).

The inversion of the generalized ray transform may therefore be done as follows. We first apply $R_J^*\Lambda$ to the data $R_J f(s,\theta)$ and then apply N^{-1} the inverse of the normal operator. The latter step can be done iteratively for instance by a conjugate gradient method.

4 Kinematic Inverse Source Problem

We now revisit the problem with an entirely different technique developed by Mukhometov and show that $R = R_J$ defined above is injective in the case where the weight $w \equiv 1$ and the curves are parameterized so that $|\dot{\gamma}| = 1$, i.e., curves are traveled along with speed equal to 1.

4.1 Transport equation

We consider a bounded domain $X \subset \mathbb{R}^2$ with smooth surface ∂X parameterized by $0 \le \tau \le T$ and points $x = S(\tau)$ with $S(0) = S(T)$ and $|\dot{S}(\tau)| = 1$.

For a point x in $\bar{X}$ and $0 \le \tau \le T$, we denote by $\tilde{\gamma}(x, \tau)$ the unique curve joining x and $S(\tau)$. For a function f supported in X, we define the curve integrals

$$g(\tau_1, \tau_2) = \int_{\tilde{\gamma}(S(\tau_1), \tau_2)} f dt, \tag{39}$$

where $dt = \sqrt{dx^2 + dy^2}$ is the Lebesgue distance measure along the curve. We thus travel along the curve with speed equal to 1.

We assume $g(\tau_1, \tau_2)$ known for all $0 \le \tau_1, \tau_2 \le T$, which corresponds to the curve integrals of f for all possible curves in the family passing through X.

The proof of injectivity of the reconstruction of f from knowledge of g is based on analyzing the following transport equation. We introduce the function

$$u(x, \tau) = \int_{\tilde{\gamma}(x,\tau)} f dt \tag{40}$$

for $x \in \bar{X}$. We denote by $\theta(x, \tau)$ the unit tangent vector to the curve $\tilde{\gamma}(x, \tau)$ at x and orientated such that

$$\theta(x, \tau) \cdot \nabla u(x, \tau) = f(x), \qquad \theta(x, \tau) = \begin{pmatrix} \cos \phi(x, \tau) \\ \sin \phi(x, \tau) \end{pmatrix}. \tag{41}$$

The latter relation is obtained by differentiating (40) with respect to arc length.

Exercise 4.1 *Check this.*

4.2 Variational form and energy estimates

We now differentiate the above with respect to τ and obtain

$$\frac{\partial}{\partial \tau}\big(\theta \cdot \nabla u\big) = 0. \tag{42}$$

We find that

$$\frac{\partial}{\partial \tau}\theta = \phi_\tau J\theta, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \phi_\tau := \partial_\tau \phi.$$

We calculate

$$J\theta \cdot \nabla u \frac{\partial}{\partial \tau}\theta \cdot \nabla u = \phi_\tau (J\theta \cdot \nabla u)^2 + J\theta \cdot \nabla u \, \theta \cdot \nabla u_\tau$$

with $u_\tau = \partial_\tau u$. Similarly, we have

$$-\theta\cdot\nabla u \frac{\partial}{\partial\tau} J\theta\cdot\nabla u = \phi_\tau(\theta\cdot\nabla u)^2 - \theta\cdot\nabla u\, J\theta\cdot\nabla u_\tau.$$

Upon adding these two identities and using (42), we obtain

$$\begin{aligned} -\frac{\partial}{\partial\tau}\Big(J\theta\cdot\nabla u\,\theta\cdot\nabla u\Big) &= \phi_\tau|\nabla u|^2 + J\theta\cdot\nabla u\,\theta\cdot\nabla u_\tau - \theta\cdot\nabla u\, J\theta\cdot\nabla u_\tau \\ &= \phi_\tau|\nabla u|^2 + \nabla\cdot(J\nabla u\, u_\tau). \end{aligned}$$

Indeed, denoting by $R=(\theta|J\theta)$ the rotation matrix and $T=(\nabla u|\nabla u_\tau)$, we find that

$$\theta\cdot aJ\theta\cdot b - J\theta\cdot a\theta\cdot b = \det(R^tT) = \det(R)\mathrm{detT} = \mathrm{detT} = Ja\cdot b.$$

independent of θ. This little miracle occurs in dimension $n=2$. For $a=\nabla u$ and $b=\nabla u_\tau$, this gives $J\nabla u\cdot\nabla u_\tau = \nabla\cdot(J\nabla u\, u_\tau)$ since $\nabla\cdot J\nabla = 0$. It remains to integrate over $X\times(0,T)$ and the fact that $S(0)=S(T)$ on the surface of X to obtain that

$$\int_0^T\int_X \phi_\tau|\nabla u|^2 dx d\tau = \int_0^T\int_{\partial X} \nabla u\cdot Jn(x,\tau)u_\tau(x,\tau)d\Sigma(x)d\tau, \tag{43}$$

where n is the outward unit normal to X at $x\in\partial X$ and $d\Sigma(x)$ the surface (length) measure on ∂X. Now, $S(\tau')$ at the surface has tangent vector $\dot S(\tau')d\tau' = -Jn(\tau')d\Sigma(x)$ assuming the parameterization $S(\tau)$ counter-clock-wise. Since $u(S(\tau'),\tau) = g(\tau',\tau)$, we find that $\nabla u\cdot\dot S(\tau') = \partial_{\tau'}g(\tau',\tau)$ and $u_\tau(x,\tau) = \partial_\tau g(\tau',\tau)$ so that eventually,

$$\int_0^T\int_X \phi_\tau(x,\tau)|\nabla u|^2(x,\tau)dxd\tau = -\int_0^T\int_0^T \frac{\partial}{\partial\tau_1}g(\tau_1,\tau_2)\frac{\partial}{\partial\tau_2}g(\tau_1,\tau_2)d\tau_1 d\tau_2. \tag{44}$$

From the definition of τ and ϕ_τ, we observe that

$$\det(\theta|\partial_\tau\theta) = \phi_\tau\det(\theta|J\theta) = \phi_\tau.$$

The **assumption** we make on the family of curves is such that the vector $\partial_\tau\theta$ cannot vanish and cannot be parallel to θ. In the choice of orientation of S, we find that

$$\phi_\tau > 0. \tag{45}$$

Note that this is a non-local assumption on the curves. It states that the curves passing by a point x separate sufficiently rapidly in the sense that ϕ increases sufficiently rapidly as the boundary parameter τ increases.

4.3 Injectivity result

Since $|f(x)| \le |\nabla u(x,\tau)|$ from the definition of the transport equation and ϕ_τ integrates to 2π in τ, we find that

$$2\pi\int_X |f(x)|^2 dx = \int_0^T\int_X \phi_\tau|f(x)|^2 dx d\tau \le -\int_0^T\int_0^T \frac{\partial}{\partial\tau_1}g(\tau_1,\tau_2)\frac{\partial}{\partial\tau_2}g(\tau_1,\tau_2)d\tau_1 d\tau_2. \tag{46}$$

Since $g(\tau, \tau') = g(\tau', \tau)$, this shows that

$$\|f\|_{L^2(X)} \leq \frac{1}{\sqrt{2\pi}} \|\partial_\tau g\|_{L^2((0,T)\times(0,T))}. \tag{47}$$

When two measurements g are equal so that their difference and hence the difference of their differential vanishes, then the difference of sources f also vanishes. This provides the injectivity of the transform $Rf(s, \theta)$ for f supported on a compact domain. Indeed, if $g = 0$, the $\partial_\tau g = 0$ and hence $f = 0$. This gives the injectivity. However, the proof of injectivity is not a proof of invertibility as we have for the normal opeartor N and is not as constructive as the result obtained for $F = I - T$.

4.4 Summary on GRT.

What have we done so far? We have defined a generalized ray transform $R_J f(s, \theta)$. We have then quickly brought our functions back into the space of positions by applying a rescaled adjoint operator $R_K^*\Lambda$. This lead to the definition of the operators $F = R_K^*\Lambda R_J$ and $N = R_J^*\Lambda R_J$. We have seen that by an appropriate choice of K, then $F = I - T$ where T is a compact operator mapping $L^2(X)$ to $H^1(X)$. However, we do not know that F is invertible in general although by perturbation we know that it is when the curves are close to the straight lines and the weight w is close to 1. Since T is compact, we know that the space of functions such that $T\psi = \psi$ is finite dimensional. But we do not know whether it is trivial.

We have then changed gears slightly and have looked at the normal operator $N = R_J^*\Lambda R_J$. Such an operator, like F, is a pseudo-differential operator. Moreover, it is invertible up to compact perturbations in the sense that $QN = I - T$ for T compact and Q another pseudo-differential operator of order 0. Here again, we do not know that 1 is not an eigenvalue of T nor that Q is invertible. However, we have seen that in some situations, R_J, and hence N, was injective by using a transport equation. This allowed us to show that N was in fact an invertible operator in $L^2(X)$. Once properly discretized, the resulting equations may then be solved by the method of, e.g., conjugate gradient.

This provides a reasonable theory for the reconstruction of functions from full data, i.e., from knowledge of $R_J f(s, \theta)$ for all $(s, \theta) \in \mathbb{R} \times \mathbb{S}^1$. In many practical problems, such data are not available. It then remains a very difficult problem to prove injectivity of the transform. In the case of the Radon transform, the Fourier slice theorem shows that the Radon transform is injective as soon as an open set of values of θ is available (and all values of s). This is because the Fourier transform of a compactly supported function is an analytic function in the Fourier variable and that an analytic function known on an arbitrarily small open set is known everywhere. For the generalized ray transform, no such results are available. However, it is interesting to understand which singularities of the function $f(x)$ may be reconstructed from available measurements. This requires that we understand how singularities propagate when we apply the Radon transform and the adjoint of the Radon transform.

5 Propagation of singularities for the GRT.

5.1 Wave Front Set and Distributions.

As we have seen, the notion of singular support is not sufficient to describe the propagation of singularities for operators that are not ΨDOs, such as for instance the Radon transform or the generalized ray transform. The singular supports have to be extended and refined to a phase space notion (the cotangent bundle). We then need a map from cotangent bundle to cotangent bundle describing how singularities propagate.

For $u \in \mathcal{D}'(X)$ and $X \subset \mathbb{R}^n$, we define the Wave Front Set of u denoted by $WF(u)$ as follows. We say that $(x_0, \xi_0) \notin WF(u)$ iif there exists a function $\phi \in C_0^\infty(X)$ with $\phi(x_0) \neq 0$ such that the Fourier transform $\widehat{\phi u}(\xi)$ is rapidly decreasing in a conic neighborhood of the half ray with direction ξ_0, i.e., for ξ such that $\hat{\xi} - \hat{\xi}_0$ is sufficiently small.

The main result on Wave Front Sets is then as follows:

Theorem 5.1 *Let $X \in \mathbb{R}^n$, Γ an open cone in $X \times (\mathbb{R}^n \backslash \{0\})$ and ϕ a phase function in Γ. If $a \in S^m(X \times \mathbb{R}^N)$, vanishes near the zero section and* $\mathrm{cone\,supp}\, a \subset \Gamma$*, then for A seen as a distribution and defined in* (20)*, we have*

$$WF(A) \subset \{(x, \phi'_x);\ (x, \zeta) \in \mathrm{cone\,supp}\, a, \quad \phi'_\zeta(x, \zeta) = 0\}. \tag{48}$$

Proof. The proof of this result goes as follows. Let $\varphi(x)$ be a function concentrating near a point x_0. Then

$$\widehat{A\varphi}(\xi) = \int_{\mathbb{R}^N \times \mathbb{R}^n} e^{i\phi(x,\zeta) - ix\cdot\xi} a(x, \zeta)\varphi(x) d\zeta dx.$$

Let $\psi(x, \zeta) = \phi(x, \zeta) - \xi$. Then

$$d\psi = \phi'_\zeta d\zeta + (\phi'_x - \xi)dx.$$

This means that for ξ in a cone away from $\xi_0 = \phi'_x$, we can define a smooth differential operator of the form $L = a \cdot \nabla_x + c$ so that $L^t e^{i(\phi(x,\xi) - x\cdot\zeta)} = e^{i(\phi(x,\xi) - x\cdot\zeta)}$. The reason is that since ξ is away from $\xi_0 = \phi'_x$, which is homogeneous of degree one in ζ, then

$$|\phi'_x(x, \zeta) - \xi| \geq C(|\zeta| + |\xi|).$$

The usual integrations by parts then give us that $|\widehat{A\varphi}(\xi)| \leq C|\xi|^{-k}$ for all $k \in \mathbb{N}$. This proves that $(x_0, \xi_0) \notin WF(A)$ and concludes the proof of the result. □

In particular, if A is a ΨDO, then $\phi(x, y, \zeta) = (x - y) \cdot \zeta$ and we find that

$$WF(K_A) \subset N^*\Delta := \{(x, x, \zeta, -\zeta),\ x \in X,\ \zeta \in \mathbb{R}^n - 0\}.$$

For more general operators, rules of propagation of singularities can also be defined. We present them without derivations, which are fairly technical.

To do so, it is instructive to look at the product of distributions and understand when such products are defined. Let Γ_j, $j = 1, 2$ be two closed cones in $X \times (\mathbb{R}^N - 0)$.

(What we mean is that they are cones in the ζ variable, not the x variable.) We *assume* that

$$\Gamma_1 + \Gamma_2 = \{(x, \zeta_1 + \zeta_2),\ (x, \zeta_j) \in \Gamma_j\} \subset X \times (\mathbb{R}^N - 0). \tag{49}$$

The "-0" above is the important information. We assume that ζ_1 and ζ_2 *cannot* be linearly dependent for an x as the base point in both cones. Then $(\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$ is also a closed cone in $X \times (\mathbb{R}^N - 0)$. We have the

Theorem 5.2 *Let Γ_j be closed cones as above. Then the product of distributions u_j such that $WF(u_j) \subset \Gamma_j$ can be defined in one and only one way so that it is sequentially continuous with values in $\mathcal{D}'$. Moreover, we have*

$$WF(u_1 u_2) \subset (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2.$$

A prototypical example is the product of δ_{x_1} and δ_{x_2} in $\mathbb{R}^n$ for $n \geq 2$. Indeed, for $n = 2$, $\Gamma_1 = \{(0, x_2, \zeta_1, 0) - 0\}$ and $\Gamma_2 = \{(x_1, 0, 0, \zeta_2) - 0\}$. Then $\Gamma_1 + \Gamma_2 = \{(0, 0, \zeta_1, \zeta_2) - 0\}$ and the above WFS inclusion is clearly satisfied for $\delta_x := \delta_{x_1}\delta_{x_2}$.

Note that in fact, $WF(\delta_x) = \Gamma_1 + \Gamma_2 = \{(0, \zeta) - 0\}$. The inclusion may thus not be an equality. The reason is that singularities of δ_{x_1} at $x_2 \neq 0$ are no longer present in the product since $u_2 = \delta_{x_2}$ vanishes there. Inclusions may therefore be strict as such cancellations do occur.

Let us look at the propagation of singularities in a linear transformation. Let a distribution $K \in \mathcal{D}'(X \times Y)$ for $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Then K defines a continuous map from $C_0^\infty(Y)$ to $\mathcal{D}'(X)$ via

$$\langle K\varphi, \psi\rangle := K(\psi \otimes \varphi); \quad \varphi \in C_0^\infty(Y),\ \psi \in C_0^\infty(X).$$

We use K both for the operator (on the left) and the distribution (on the right).

Let $u \in C_0^\infty(Y)$. Then

$$WF(Ku) \subset WF_X(K) := \{(x, \xi); (x, \xi, y, 0) \in WF(K)\}.$$

In other words, if K has no singularities that are purely in (x, ξ), then $Ku \in C^\infty(X)$.

The definition of Ku for u a distribution is a dual question. Let $u_1 = K$ the distribution on $X \times Y$. Let $u_2 = 1 \otimes u$ another distribution. The product is well defined if the cones $\Gamma_1 = WF(K)$ and $\Gamma_2 = WF(1 \otimes u) = X \times WF(u)$ are such that the sum $\Gamma_1 + \Gamma_2$ satisfies (49), i.e., does not meet $\zeta_1 + \zeta_2 = 0$. This means that $WF(u)$ does not meet

$$WF'_Y(K) = \{(y, \eta);\ (x, 0, y, -\eta) \in WF(K) \text{ for some } x\ \}.$$

When $WF'_Y(K) = \emptyset$, then K defines a continuous map from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.

5.2 Propagation of singularities in FIOs

With this, we arrive at the main result:

Theorem 5.3 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ and $K \in \mathcal{D}'(X \times Y)$. If $WF_X(K) = \emptyset$ and for $u \in \mathcal{E}'(Y)$, $WF(u)$ does not meet $WF'_Y(K)$, then*

$$WF(Ku) \subset WF'(K)WF(u), \tag{50}$$

where we have defined

$$WF'(K) = \{(x, \xi, y, -\eta) \in X \times \mathbb{R}^n \times Y \times \mathbb{R}^m;\ (x, \xi, y, \eta) \in WF(K)\} \qquad (51)$$

where $WF'(K)$ is regarded as a relation mapping sets in $Y \times \mathbb{R}^m - 0$ to sets in $X \times \mathbb{R}^n - 0$.

In other words, $(x, \xi) \in WF'(K)WF(u)$ when there exists $(y, \eta) \in WF(u)$ such that $(x, \xi, y, -\eta) \in WF(K)$, the latter being equivalent to $(x, \xi, y, \eta) \in WF'(K)$.

For $K = \delta(x - y) = c_n \int e^{i(x-y)\cdot\xi} d\xi$, which corresponds to the phase $\phi(x, y, \xi) = (x - y) \cdot \xi$, we have seen that $WF'(K) = \{(x, x, \xi, \xi),\ (x, \xi) \in X \times \mathbb{R}^n - 0\}$. Thus, $WF(Ku) \subset WF(u)$. More generally, for any distribution of a ΨDO, we have

$$K(x, y) = \int e^{i(x-y)\cdot\xi} a(x, y, \xi) d\xi,$$

with also $WF'(K) = \{(x, x, \xi, \xi),\ (x, \xi) \in X \times \mathbb{R}^n - 0\}$. This shows that ΨDOs do not propagate singularities. This is a refined version of the statement we had regarding the singular support of a distribution.

Let us now consider the phase $\phi(s, \theta, x, \sigma) = (s - s(x, \theta))\sigma$ and the distribution kernel

$$K(s, \theta, x) = \int_{\mathbb{R}} e^{i(s-s(x,\theta))\sigma} \frac{J(x, \theta)}{2\pi} d\sigma.$$

The phase stations at $s = s(x, \theta)$, the set where $\phi'_\sigma = 0$. Away from that set, the distribution kernel is smooth. So all the action takes place on the set

$$WF(K) = \Big\{ \big(s, \theta, x, \sigma, -\sigma\partial_\theta s(x, \theta), -\sigma\nabla_x s(x, \theta)\big),\ s = s(x, \theta) \Big\}.$$

This implies that

$$WF'(K) = \Big\{ \big(s(x, \theta), \theta, x, \sigma, -\sigma\partial_\theta s(x, \theta), \sigma\nabla_x s(x, \theta)\big) \Big\}.$$

How are singularities propagated by the generalized ray transform then? Let us assume we have a singularity of $f(x)$ at $(x, \xi) \in X \times \mathbb{R}^2 - 0$. Then in order for that singularity to propagate to the Radon domain, it has to be of the form

$$\xi = \sigma\nabla_x s(x, \theta).$$

Our hypotheses on s show that $(\sigma, \theta) \to \xi$ is in fact a diffeomorphism of $\mathbb{R}_+ \times \mathbb{S}_1$ to $\mathbb{R}^2 - 0$ at each x. Therefore, θ and σ are uniquely determined by ξ. This means that the singularity at (x, ξ) will propagate to a singularity at

$$(s(x, \theta), \theta, \sigma) \in \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}_+ - 0.$$

Note that the singularity in the latter space is uniquely defined. This should be contrasted to the very different result we obtained for the singular supports, stating that a singularity at x may propagate into singularities for all $(s(x, \theta), \theta)$ with $\theta \in \mathbb{S}^1$.

We have to worry about a slight additional complication here. The above singularity in the phase space of the Radon domain is uniquely defined for $\sigma > 0$. It is also uniquely defined for $\sigma < 0$. So it turns out that a singularity at (x, ξ) actually propagates into two singularities in $(s,\theta,\xi_s,\xi_\theta)$. This is related to the fact that the parameterization $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$ is a double covering of the space of lines: a line parameterized by (s,θ) is the same line as the one parameterized by $(-s,-\theta)$. The two corresponding curves are different in general. But we also see in the setting of integrals along curves that one singularity propagates to two places.

At any rate, we observe that the notion of Wave Front Set is crucial to our understanding of the propagation of singularities in generalized ray transforms.

What happens to the adjoint operator R_J^*? Its distribution kernel is

$$K'(x,s,\theta)=\int_{\mathbb{R}} e^{i(s-s(x,\theta))\sigma}\frac{J(x,\theta)}{2\pi}d\sigma.$$

Therefore,

$$WF'(K')=\Big\{\big(x,s(x,\theta),\theta,-\sigma\nabla_x s(x,\theta),-\sigma,\sigma\partial_\theta s(x,\theta)\big)\Big\}.$$

Let us now assume that we have a singularity at point $(s,\theta,\zeta_s,\zeta_\theta)$. We have seen that $(s(x,\theta),\partial_\theta s(x,\theta))$ uniquely characterizes x. This is a property of the curves. This implies that x is uniquely determined by knowledge of $(s,\theta,\zeta_s,\zeta_\theta)$. And so is $\xi=-\sigma\nabla_x s(x,\theta)$. We thus obtain again that singularities in the Radon domain corresponded to uniquely determined singularities in the spatial domain. Moreover, the two singularities generated by a single (x,ξ) both back-propagate to (x,ξ) by composition with $WF'(K')$.

We can therefore apply the operator $R_J^*R_J$, and observe by composition of the previous two propagations that a singularity at (x,ξ) is mapped into the same singularity at (x,ξ). This is consistent with the fact that $R_J^*R_J$ is a pseudo-differential operator with a phase that can be recast as $(x-y)\cdot\xi$.

In the above discussion, the phase ϕ was seen as the main player responsible for the propagation of singularities in $X\times\mathbb{R}^n-0$ to singularities in $Y\times\mathbb{R}^m-0$. However, does such transfer always occur? This now depends on the amplitude $a(x,y,\zeta)$. Let us recall the form of the kernel

$$K(x,y)=\int e^{i\phi(x,y,\zeta)}a(x,y,\zeta)d\zeta.$$

ϕ is a function in a $(n+m+N)-$dimensional manifold. Then (x,y,ϕ'_x,ϕ'_y) is defined in a $(n+m+N)-$dimensional manifold. However, we restrict ourselves to the submanifold where $\phi'_\zeta=0$, which imposes N constraints. Therefore, $\{(x,y,\phi'_x,\phi'_y),\ \phi'_\zeta=0\}$ a priori is defined in a $(n+m)-$dimensional manifold. This is half the dimension of the product space $X\times\mathbb{R}^n\times Y\times\mathbb{R}^m$. This forms what is called a Lagrangian manifold. And if we want a one-to-one correspondence between singularities (or a one-to-two correspondence as we have seen is the case for the ray transform), then $m=n$. For our purpose, this means that once a singularity is known at $(y_0,\xi=\phi'(y_0))$, then this also defines x_0 and ζ_0 on that Lagrangian manifold.

We now want to know whether $a(x_0,y_0,\zeta_0)$ vanishes or not. If it vanishes, then the singularity does not propagate to the x variables. If $a(x_0,y_0,\zeta_0)\neq 0$, then the singularity propagates to the x variables. Moreover, the strength of $a(x_0,y_0,\zeta_0)$ tells us how the singularities are attenuated (or increased, though rarely in inverse problems) by the transform.

There is no simple explanation for the amount of smoothing obtained by FIOs. As a general procedure, let n be the dimension of X and Y and let N be the dimension of the fiber variable ζ. For the GRT, this is $n = 2$ and $N = 1$. Then we define a symbol $a(x, y, \zeta)$ as an element in $S^{m+\frac{1}{4}(2n-N)}(X \times Y \times \mathbb{R}^N)$. This strange normalization means that a symbol that looks non-smoothing with $m + \frac{1}{4}(2n - 2N) = 0$ as we have in the GRT, in fact is an operator smoothing by $-m$ derivatives with $m = \frac{1}{2}(N - n)$, which is $m = -\frac{1}{2}$ for the Radon transform. This is the reason why we have to multiply the symbol $J(x, \theta)$ by $\sqrt{|\sigma|}$ to obtain an operator of order 0 (i.e., bounded from L^2 to L^2).

So, to summarize this discussion, we have that the phase $\phi(x, y, \zeta)$ dictates how singularities propagate from (y, η) to (x, ξ) via the canonical relation $(x, \phi'_x, y, -\phi'_y)$ defined on the manifold $\phi'_\zeta = 0$. On that manifold, for (y, η) known, then ζ is also known so that (x, y, ζ) is known and thus so is $a(x, y, \zeta)$. When the latter does not vanish, then the singularity does propagate. When $a(x, y, \zeta)$ is strictly positive for all such points C_ϕ, then we say that the FIO is *elliptic*. We are then guaranteed that all singularities will propagate. These singularities are then typically attenuated in the sense that the amplitude of the singularity is multiplied by a constant times $|\zeta|^m$ (asymptotically for large ζ). What the value of m is is dictated by $a(x, y, \zeta)$, which is a symbol in $S^{m+\frac{n}{2}-\frac{N}{4}}$. Again, for the GRT, this means that $m = -\frac{1}{2}$. Singularities are attenuated by $|\zeta|^{\frac{1}{2}}$ by the GRT and then attenuated by $|\zeta|^{\frac{1}{2}}$ by the adjoint transform. Therefore, $R_J^* R_J$ has to be multiplied by $(-\Delta)^{\frac{1}{2}}$ or replaced by $R_J^* \Lambda R_J$ in order to define an operator that is bounded from L^2 to L^2 and in which singularities are neither amplified nor attenuated.